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The integrals in Gradshteyn and Ryzhik.
Part 4: The gamma function

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Abstract. We present a systematic derivation of some definite integrals in the classical table of Gradshteyn and Ryzhik that can be reduced to the gamma function.

1. Introduction

The table of integrals [**2**] contains some evaluations that can be derived by elementary means from the *gamma function*, defined by

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \, dx. \tag{1.1}$$

The convergence of the integral in (1.1) requires $a > 0$. The goal of this paper is to present some of these evaluations in a systematic manner. The techniques developed here will be employed in future publications. The reader will find in [**1**] analytic information about this important function.

The gamma function represents the extension of factorials to real parameters. The value

$$\Gamma(n) = (n-1)!, \text{ for } n \in \mathbb{N} \tag{1.2}$$

is elementary. On the other hand, the special value

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \tag{1.3}$$

is equivalent to the well-known *normal integral*

$$\int_0^\infty \exp(-t^2) \, dt = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right). \tag{1.4}$$

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The reader will find in [**1**] proofs of Legendre's duplication formula

$$\Gamma\left(x+\tfrac{1}{2}\right) = \frac{\Gamma(2x)\sqrt{\pi}}{\Gamma(x)\,2^{2x-1}}, \tag{1.5}$$

that produces for $x = m \in \mathbb{N}$ the values

$$\Gamma\left(m+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m}}\,\frac{(2m)!}{m!}. \tag{1.6}$$

This appears as **3.371** in [**2**].

2. The introduction of a parameter

The presence of a parameter in a definite integral provides great amount of flexibility. The change of variables $x = \mu t$ in (1.1) yields

$$\Gamma(a) = \mu^a \int_0^\infty t^{a-1} e^{-\mu t}\,dt. \tag{2.1}$$

This appears as **3.381.4** in [**2**] and the choice $a = n+1$, with $n \in \mathbb{N}$, that reads

$$\int_0^\infty t^n e^{-\mu t}\,dt = n!\,\mu^{-n-1} \tag{2.2}$$

appears as **3.351.3**.

The special case $a = m + \frac{1}{2}$, that appears as **3.371** in [**2**], yields

$$\int_0^\infty t^{m-\frac{1}{2}} e^{-\mu t}\,dt = \frac{\sqrt{\pi}}{2^{2m}}\,\frac{(2m)!}{m!}\mu^{-m-\frac{1}{2}}, \tag{2.3}$$

is consistent with (1.6).

The combination

$$\int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{x^{\rho+1}}\,dx = \frac{\mu^\rho - \nu^\rho}{\rho}\,\Gamma(1-\rho), \tag{2.4}$$

that appears as **3.434.1** in [**2**] can now be evaluated directly. The parameters are restricted by convergence: μ, $\nu > 0$ and $\rho < 1$. The integral **3.434.2**

$$\int_0^\infty \frac{e^{-\mu x} - e^{-\nu x}}{x}\,dx = \ln\frac{\nu}{\mu}, \tag{2.5}$$

is obtained from (2.4) by passing to the limit as $\rho \to 0$. This is an example of *Frullani integrals* that will be discussed in a future publication.

The reader will be able to check **3.478.1**:

$$\int_0^\infty x^{\nu-1}\exp(-\mu x^p)\,dx = \frac{1}{p}\mu^{-\nu/p}\Gamma\left(\frac{\nu}{p}\right), \tag{2.6}$$

and **3.478.2**:

$$\int_0^\infty x^{\nu-1}\left[1-\exp(-\mu x^p)\right]\,dx = -\frac{1}{|p|}\mu^{-\nu/p}\Gamma\left(\frac{\nu}{p}\right) \tag{2.7}$$

by introducing appropriate parameter reduction.

The parameters can be used to prove many of the classical identities for $\Gamma(a)$.

Proposition 2.1. The gamma function satisfies

$$\Gamma(a+1) = a\,\Gamma(a). \tag{2.8}$$

Proof. Differentiate (2.1) with respect to μ to produce

$$0 = a\mu^{a-1}\int_0^\infty t^{a-1}e^{-\mu t}\,dt - \mu^a\int_0^\infty t^a e^{-\mu t}\,dt. \tag{2.9}$$

Now put $\mu = 1$ to obtain the result. ☐

Differentiating (1.1) with respect to the parameter a yields

$$\Gamma'(a) = \int_0^\infty x^{a-1}e^{-x}\,\ln x\,dx. \tag{2.10}$$

Further differentiation introduces higher powers of $\ln x$:

$$\Gamma^{(n)}(a) = \int_0^\infty x^{a-1}e^{-x}\,(\ln x)^n\,dx. \tag{2.11}$$

In particular, for $a = 1$, we obtain:

$$\int_0^\infty (\ln x)^n\,e^{-x}\,dx = \Gamma^{(n)}(1). \tag{2.12}$$

The special case $n = 1$ yields

$$\int_0^\infty e^{-x}\,\ln x\,dx = \Gamma'(1). \tag{2.13}$$

The reader will find in [**1**], page 176 an elementary proof that $\Gamma'(1) = -\gamma$, where

$$\gamma := \lim_{n\to\infty}\sum_{k=1}^n \frac{1}{k} - \ln n \tag{2.14}$$

is Euler's constant. This is one of the fundamental numbers of Analysis.

On the other hand, differentiating (2.1) produces

$$\int_0^\infty x^{a-1}e^{-\mu x}\,(\ln x)^n\,dx = \left(\frac{\partial}{\partial a}\right)^n\left[\mu^{-a}\Gamma(a)\right], \tag{2.15}$$

that appears as **4.358.5** in [**2**]. Using Leibnitz's differentiation formula we obtain

$$\int_0^\infty x^{a-1}e^{-\mu x}\,(\ln x)^n\,dx = \mu^{-a}\sum_{k=0}^n(-1)^k\binom{n}{k}(\ln\mu)^k\,\Gamma^{(n-k)}(a). \tag{2.16}$$

In the special case $a = 1$ we obtain

$$\int_0^\infty e^{-\mu x} (\ln x)^n \, dx = \frac{1}{\mu} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\ln \mu)^k \, \Gamma^{(n-k)}(1). \tag{2.17}$$

The cases $n = 1, 2, 3$ appear as **4.331.1**, **4.335.1** and **4.335.3** respectively.

In order to obtain analytic expressions for the terms $\Gamma^{(n)}(1)$, it is convenient to introduce the *polygamma function*

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x). \tag{2.18}$$

The derivatives of ψ satisfy

$$\psi^{(n)}(x) = (-1)^{n+1} n! \, \zeta(n+1, x), \tag{2.19}$$

where

$$\zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z} \tag{2.20}$$

is the *Hurwitz zeta function.* In particular this gives

$$\psi^{(n)}(1) = (-1)^{n+1} \, n! \, \zeta(n+1). \tag{2.21}$$

The values of $\Gamma^{(n)}(1)$ can now be computed by recurrence via

$$\Gamma^{(n+1)}(1) = \sum_{k=0}^{n} \binom{n}{k} \Gamma^{(k)}(1) \psi^{(n-k)}(1), \tag{2.22}$$

obtained by differentiating $\Gamma'(x) = \psi(x)\Gamma(x)$.

Using (2.19) the reader will be able to check the first few cases of (2.15), we employ the notation $\delta = \psi(a) - \ln \mu$:

$$\begin{aligned}
\int_0^\infty x^{a-1} e^{-\mu x} \ln^2 x \, dx &= \frac{\Gamma(a)}{\mu^a} \left\{ \delta^2 + \zeta(2, a) \right\}, \\
\int_0^\infty x^{a-1} e^{-\mu x} \ln^3 x \, dx &= \frac{\Gamma(a)}{\mu^a} \left\{ \delta^3 + 3\zeta(2, a)\delta - 2\zeta(3, a) \right\}, \\
\int_0^\infty x^{a-1} e^{-\mu x} \ln^4 x \, dx &= \frac{\Gamma(a)}{\mu^a} \left\{ \delta^4 + 6\zeta(2, a)\delta^2 - 8\zeta(3, a)\delta + 3\zeta^2(2, a) + 6\zeta(4, a) \right\}.
\end{aligned}$$

These appear as **4.358.2**, **4.358.3** and **4.358.4**, respectively.

3. Elementary changes of variables

The use of appropriate changes of variables yields, from the basic definition (1.1), the evaluation of more complicated definite integrals. For example, let $x = t^b$ to obtain, with $c = ab - 1$,

$$\int_0^\infty t^c \exp(-t^b) \, dt = \frac{1}{b} \Gamma\left(\frac{c+1}{b} \right). \tag{3.1}$$

The special case $a = 1/b$, that is $c = 0$, is

$$\int_0^\infty \exp(-t^b)\,dt = \frac{1}{b}\Gamma\left(\frac{1}{b}\right), \tag{3.2}$$

that appears as **3.326.1** in [**2**]. The special case $b = 2$ is the normal integral (1.4).

We can now introduce an extra parameter via $t = s^{1/b}x$. This produces

$$\int_0^\infty x^m \exp(-sx^b)\,dx = \frac{\Gamma(a)}{s^a b}, \tag{3.3}$$

with $m = ab - 1$. This formula appears (at least) three times in [**2**]: **3.326.2**, **3.462.9** and **3.478.1**. Moreover, the case $s = 1$, $c = (m + 1/2)n - 1$ and $b = n$ appears as **3.473**:

$$\int_0^\infty \exp(-x^n)x^{\left(m+\frac{1}{2}\right)n-1}\,dx = \frac{(2m-1)!!}{2^m\,n}\sqrt{\pi}. \tag{3.4}$$

The form given here can be established using (1.6).

Differentiating (3.3) with respect to the parameter m (keeping in mind that $a = (m+1)/b$), yields

$$\int_0^\infty x^m e^{-sx^b}\,\ln x\,dx = \frac{\Gamma(a)}{b^2\,s^a}\left[\psi(a) - \ln s\right]. \tag{3.5}$$

In particular, if $b = 1$ we obtain

$$\int_0^\infty x^m e^{-sx}\,\ln x\,dx = \frac{\Gamma(m+1)}{s^{m+1}}\left[\psi(m+1) - \ln s\right]. \tag{3.6}$$

The case $m = 0$ and $b = 2$ gives

$$\int_0^\infty e^{-sx^2}\,\ln x\,dx = -\frac{\sqrt{\pi}}{4\sqrt{s}}\left(\gamma + \ln 4s\right), \tag{3.7}$$

where we have used $\psi(1/2) = -\gamma - 2\ln 2$. This appears as **4.333** in [**2**].

An interesting example is $b = m = 2$. Using the values

$$\Gamma\left(\tfrac{3}{2}\right) = \sqrt{\pi}/2 \text{ and } \psi\left(\tfrac{3}{2}\right) = 2 - 2\ln 2 - \gamma \tag{3.8}$$

the expression (3.5) yields

$$\int_0^\infty x^2 e^{-sx^2}\,\ln x\,dx = \frac{1}{8s}(2 - \ln 4s - \gamma)\sqrt{\frac{\pi}{s}}. \tag{3.9}$$

The values of ψ at half-integers follow directly from (1.5). Formula (3.9) appears as **4.355.1** in [**2**]. Using (3.5) it is easy to verify

$$\int_0^\infty (\mu x^2 - n)x^{2n-1}e^{-\mu x^2}\,\ln x\,dx = \frac{(n-1)!}{4\mu^n}, \tag{3.10}$$

and

$$\int_0^\infty (2\mu x^2 - 2n - 1)x^{2n}e^{-\mu x^2}\,\ln x\,dx = \frac{(2n-1)!!}{2(2\mu)^n}\sqrt{\frac{\pi}{\mu}}, \tag{3.11}$$

for $n \in \mathbb{N}$. These appear as, respectively, **4.355.3** and **4.355.4** in [**2**]. The term $(2n-1)!!$ is the semi-factorial defined by

$$(2n-1)!! = (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1. \tag{3.12}$$

Finally, formula **4.369.1** in [**2**]

$$\int_0^\infty x^{a-1}e^{-\mu x}\left[\psi(a)-\ln x\right]\,dx = \frac{\Gamma(a)\,\ln\mu}{\mu^a} \tag{3.13}$$

can be established by the methods developed here. The more ambitious reader will check that

$$\int_0^\infty x^{n-1}e^{-\mu x}\left\{[\ln x - \tfrac{1}{2}\psi(n)]^2 - \tfrac{1}{2}\psi'(n)\right\}\,dx = \frac{(n-1)!}{\mu^n}\left\{[\ln\mu - \tfrac{1}{2}\psi(n)]^2 + \tfrac{1}{2}\psi'(n)\right\},$$

that is **4.369.2** in [**2**].

We can also write (3.5) in the exponential scale to obtain

$$\int_{-\infty}^\infty te^{mt}\exp\left(-se^{bt}\right)\,dt = \frac{\Gamma(m/b)}{b^2s^{m/b}}\left(\psi\left(\frac{m}{b}\right)-\ln s\right). \tag{3.14}$$

The special case $b=m=1$ produces

$$\int_{-\infty}^\infty te^t\exp\left(-se^t\right)\,dt = -\frac{(\gamma+\ln s)}{s} \tag{3.15}$$

that appears as **3.481.1**. The second special case, appearing as **3.481.2**, is $b=2$, $m=1$, that yields

$$\int_{-\infty}^\infty te^t\exp\left(-se^{2t}\right)\,dt = -\frac{\sqrt{\pi}\,(\gamma+\ln 4s)}{4\sqrt{s}}. \tag{3.16}$$

This uses the value $\psi(1/2) = -(\gamma+2\ln 2)$.

There are many other possible changes of variables that lead to interesting evaluations. We conclude this section with one more: let $x=e^t$ to convert (1.1) into

$$\int_{-\infty}^\infty \exp\left(-e^x\right)\,e^{ax}\,dx = \Gamma(a). \tag{3.17}$$

This is **3.328** in [**2**].

As usual one should not prejudge the difficulty of a problem: the example **3.471.3** states that

$$\int_0^a x^{-\mu-1}(a-x)^{\mu-1}e^{-\beta/x}\,dx = \beta^{-\mu}a^{\mu-1}\Gamma(\mu)\,\exp\left(-\tfrac{\beta}{a}\right). \tag{3.18}$$

This can be reduced to the basic formula for the gamma function. Indeed, the change of variables $t=\beta/x$ produces

$$I = \beta^{-\mu}a^{\mu-1}\int_{\beta/a}^\infty (t-\beta/a)^{\mu-1}\,e^{-t}\,dt. \tag{3.19}$$

Now let $y = t - \beta/a$ to complete the evaluation. The table [**2**] writes μ instead of a: it seems to be a bad idea to have μ and u in the same formula, it leads to typographical errors that should be avoided.

Another simple change of variables gives the evaluation of **3.324.2**:

$$\int_{-\infty}^{\infty} e^{-(x-b/x)^{2n}}\, dx = \frac{1}{n}\Gamma\left(\frac{1}{2n}\right). \tag{3.20}$$

The symmetry yields

$$I = 2\int_0^{\infty} e^{-(x-b/x)^{2n}}\, dx. \tag{3.21}$$

The change of variables $t = b/x$ yields, using $b > 0$,

$$I = 2b\int_0^{\infty} e^{-(t-b/t)^{2n}}\, \frac{dt}{t^2}. \tag{3.22}$$

The average of these forms produces

$$I = \int_0^{\infty} e^{-(x-b/x)^{2n}} \left(1 + \frac{b}{x^2}\right)\, dx. \tag{3.23}$$

Finally, the change of variables $u = x - b/x$ gives the result. Indeed, let $u = x - b/x$ and observe that u is increasing when $b > 0$. This restriction is missing in the table. Then we get

$$I = 2\int_0^{\infty} e^{-u^{2n}}\, du. \tag{3.24}$$

This can now be evaluated via $v = u^{2n}$.

Note. In the case $b < 0$ the change of variables $u = x - b/x$ has an inverse with two branches, splitting at $x = \sqrt{-b}$. Then we write

$$\begin{aligned} I &:= 2\int_0^{\infty} e^{-(x-b/x)^{2n}}\, dx \\ &= 2\int_0^{\sqrt{-b}} e^{-(x-b/x)^{2n}}\, dx + 2\int_{\sqrt{-b}}^{\infty} e^{-(x-b/x)^{2n}}\, dx. \end{aligned} \tag{3.25}$$

The change of variables $u = x - b/x$ is now used in each of the integrals to produce

$$I = 2\int_{2\sqrt{-b}}^{\infty} \frac{u\exp(-u^{2n})\, du}{\sqrt{u^2+4b}}. \tag{3.26}$$

The change of variables $z = \sqrt{u^2+4b}$ yields

$$I = 2\int_0^{\infty} \exp\left(-(z^2-4b)^n\right). \tag{3.27}$$

We are unable to simplify it any further.

4. The logarithmic scale

Euler prefered the version

$$\Gamma(a) = \int_0^1 \left(\ln \frac{1}{u} \right)^{a-1} du. \tag{4.1}$$

We will write this as

$$\Gamma(a) = \int_0^1 (-\ln u)^{a-1} \, du, \tag{4.2}$$

for better spacing. Many of the evaluations in [**2**] follow this form. Section **4.215** in [**2**] consists of four examples: the first one, **4.215.1** is (4.1) itself. The second one, labeled **4.215.2** and written as

$$\int_0^1 \frac{dx}{(-\ln x)^{\mu}} = \frac{\pi}{\Gamma(\mu)} \text{cosec } \mu\pi, \tag{4.3}$$

is evaluated as $\Gamma(1-\mu)$ by (4.1). The identity

$$\Gamma(\mu)\Gamma(1-\mu) = \frac{\pi}{\sin \pi\mu} \tag{4.4}$$

yields the given form. The reader will find in [**1**] a proof of this identity. The section concludes with the special values

$$\int_0^1 \sqrt{-\ln x} \, dx = \frac{\sqrt{\pi}}{2}, \tag{4.5}$$

as **4.215.3** and **4.215.4**:

$$\int_0^1 \frac{dx}{\sqrt{-\ln x}} = \sqrt{\pi}. \tag{4.6}$$

Both of them are special cases of (4.1).

The reader should check the evaluations **4.269.3**:

$$\int_0^1 x^{p-1} \sqrt{-\ln x} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{p^3}}, \tag{4.7}$$

and **4.269.4**:

$$\int_0^1 \frac{x^{p-1} \, dx}{\sqrt{-\ln x}} = \sqrt{\frac{\pi}{p}} \tag{4.8}$$

by reducing them to (2.1). Also **4.272.5**, **4.272.6** and **4.272.7**

$$\begin{aligned} \int_1^\infty (\ln x)^p \, \frac{dx}{x^2} &= \Gamma(1+p), \\ \int_0^1 (-\ln x)^{\mu-1} \, x^{\nu-1} \, dx &= \frac{1}{\nu^{\mu}} \Gamma(\mu), \\ \int_0^1 (-\ln x)^{n-\frac{1}{2}} \, x^{\nu-1} \, dx &= \frac{(2n-1)!!}{(2\nu)^n} \sqrt{\frac{\pi}{\nu}}, \end{aligned} \tag{4.9}$$

can be evaluated directly in terms of the gamma function.

Differentiating (4.1) with respect to a yields **4.229.4** in [**2**]:

$$\int_0^1 \ln(-\ln x)\ (-\ln x)^{a-1}\ dx = \Gamma'(a) = \psi(a)\Gamma(a), \tag{4.10}$$

with $\psi(a)$ defined in (2.18). The special case $a=1$ is **4.229.1**:

$$\int_0^1 \ln(-\ln x)\ dx = -\gamma, \tag{4.11}$$

and

$$\int_0^1 \ln(-\ln x)\ \frac{dx}{\sqrt{-\ln x}} = -(\gamma + 2\ln 2)\sqrt{\pi}, \tag{4.12}$$

that appears as **4.229.3**, is obtained by using the values $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\psi\left(\frac{1}{2}\right) = -(\gamma + 2\ln 2)$.

The same type of arguments confirms **4.325.11**

$$\int_0^1 \ln(-\ln x)\ \frac{x^{\mu-1}\,dx}{\sqrt{-\ln x}} = -(\gamma + \ln 4\mu)\sqrt{\frac{\pi}{\mu}}, \tag{4.13}$$

and **4.325.12**:

$$\int_0^1 \ln(-\ln x)\ (-\ln x)^{\mu-1}\ x^{\nu-1}\,dx = \frac{1}{\nu^{\mu}}\Gamma(\mu)\left[\psi(\mu) - \ln\nu\right]. \tag{4.14}$$

In particular, when $\mu = 1$ we obtain **4.325.8**:

$$\int_0^1 \ln(-\ln x)\, x^{\nu-1}\,dx = -\frac{1}{\nu}\left(\gamma + \ln\nu\right). \tag{4.15}$$

5. The presence of fake parameters

There are many formulas in [**2**] that contain parameters. For example, **3.461.2** states that

$$\int_0^\infty x^{2n}e^{-px^2}\,dx = \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}} \tag{5.1}$$

and **3.461.3** states that

$$\int_0^\infty x^{2n+1}e^{-px^2}\,dx = \frac{n!}{2p^{n+1}}. \tag{5.2}$$

The change of variables $t = px^2$ eliminates the *fake* parameter p and reduces **3.461.2** to

$$\int_0^\infty t^{n-\frac{1}{2}}e^{-t}\,dt = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \tag{5.3}$$

and **3.461.3** to

$$\int_0^\infty t^n e^{-t}\,dt = n!. \tag{5.4}$$

These are now evaluated by identifying them with $\Gamma(n+\frac{1}{2})$ and $\Gamma(n+1)$, respectively.

A second way to introduce fake parameters is to shift the integral (2.1) via $s = t+b$ to produce

$$\int_b^\infty (s-b)^{a-1} e^{-s\mu}\, ds = \mu^{-a} e^{-\mu b} \Gamma(a). \tag{5.5}$$

This appears as **3.382.2** in [**2**].

There are many more integrals in [**2**] that can be reduced to the gamma function. These will be reported in a future publication.

References

[1] G. Boros and V. Moll. *Irresistible Integrals.* Cambridge University Press, New York, 1st edition, 2004.

[2] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products.* Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition, 2007.

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