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Maximum Principles For Some Elliptic Problems

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Abstract. In this paper we introduce a maximum principle for some semilinear elliptic equations subject to mixed boundary conditions which may be used to deduce bounds on important quantities in physical problems of iterest.

1. Introduction

In [6],maximum principles for the functions $p = g(u)\left|\nabla u\right|^2 + h(u)$ and $q = g(u)\left|\nabla u\right|^2 + c\int\limits_0^u f(s)g(s)ds, c \in R$, which are defined on solutions of the semilinear partial differential equation $\Delta u + f(u) = 0$ in some region $\Omega \subset R^n$ are found using the classical maximum principle [3]. In [7], a maximum principle for the function q at a critical point of u under some conditions on $\partial\Omega$ is introduced. In [2], the following result is proved : Let $u \in C^3(\Omega)$ be a solution of

$$\Delta u + f(u) = 0 \quad in\ \Omega \subset R^n,$$

$$u = 0 \quad on\ \partial\Omega.$$

If the boudary $\partial\Omega$ has a nonnegative mean curvature, then the function

$$\Phi = \left|\nabla u\right|^2 + 2\int\limits_0^u f(s)g(s)ds$$

assumes its maximum at a point where $\nabla u = 0$. In [3], maximum principles are derived for certain functions defined for solutions of the equation

$$\Delta u + \lambda\rho(x)f(u) = 0$$

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in some region $\Omega \subset R^2$ subject to a mixed boundary condition.

In this paper we derive maximum principles for functions defined for solutions of the semilinear equation

$$\Delta u + f(x,u) = 0 \tag{1.1}$$

in some region $\Omega \subset R^n$ subject to a mixed boudary condition.

In order to motivate our work, let us first look at the one dimensional problem

$$u_{xx} + f(x,u) = 0. \tag{1.2}$$

If we multiply (1.2) by u_x we get

$$\frac{1}{2}(u_x^2)_x + f(x,u)u_x = 0,$$

that is

$$\frac{1}{2}u_x^2 + \int_0^u f(x,s)ds - H(x,u) = constant, \tag{1.3}$$

where $H(x,u)$ satisfies:

$$H_x(x,u) = \int_0^u f_x(x,s)ds.$$

Thus we conclude that the function

$$p = u_x^2 + 2\int_0^u f(x,s)ds - 2H(x,u) \tag{1.4}$$

is a constant, where u is a solution of (1.2). It is obvious that p satisfies a maximum principle.

Let u be a solution of (1.1) . We look for a function p of the form

$$p = |\nabla u|^2 + 2\int_0^u f(x,s)ds - 2H(x,u), \tag{1.5}$$

where $H(x,u)$ satisfies:

$$H_{,i(x,u)} = \int_0^u f_{,i}(x,s)ds.$$

Our goal is to find conditions such that (1.5) satisfies a maximum principle.

Let us first give the following lemma.

Lemma. *Let u be a $C^3(\overline{\Omega})$ solution of (1.1) with $f \in C^1(\Omega \times R), \Omega \subset R^n$, $n \geqslant 2$. Then the function p defined by (1.5) takes its maximum either on $\partial\Omega$ or at a critical point of u.*

Proof. By differentiating (1.5) we obtain

$$p_{,i} = 2u_{,j}u_{,ij} + 2fu_{,i} \tag{1.6}$$

$$\Delta p = p_{,ii} = 2u_{,ij}u_{,ij} + 2u_{,j}u_{,iij} + 2f\Delta u + 2f_{,i}u_{,i}. \tag{1.7}$$

Now we have

$$\Delta u = -f, \tag{1.8}$$

$$u_{,iij} = -f_{,j}. \tag{1.9}$$

This allows us to rewrite (1.7) as

$$\Delta p = 2u_{,ij}u_{,ij} - 2f^2. \tag{1.10}$$

From (1.6) and Schwarz's inequality, it follows that

$$(p_{,i} - 2fu_{,i})(p_{,i} - 2fu_{,i}) = 4u_{,ji}u_{,j}u_{,ki}u_{,k} \leqslant 4u_{,ij}u_{,ij}\left|\nabla u\right|^2. \tag{1.11}$$

Consequently, by (1.10) and (1.11) , we can write

$$\Delta p + \frac{L_k p_{,k}}{\left|\nabla u\right|^2} \geqslant 0, \tag{1.12}$$

where

$$L_k = 2fu_{,k} - \frac{1}{2}p_{,k}.$$

Hopf's first maximum principle [4] implies the lemma.

2. The Result and its Proof

We give our result by the following theorem.

Theorem. *Let u be a $C^3(\overline{\Omega})$ solution of the problem*

$$\Delta u + f(x,u) = 0 \quad in \quad \Omega,$$

$$u = 0 \quad on \quad \Gamma_1, \ \frac{\partial u}{\partial n} = 0, \quad on \quad \Gamma_2, \ \Gamma_1 \cup \Gamma_2 = \partial\Omega,$$

where $f \in C^1(\Omega \times R)$, Ω *is a convex domain in* R^2 *and* $\frac{\partial u}{\partial n}$ *denotes the outward normal derivative. Then the function* p *defined by (1.5) takes its maximum at a critical point of* u.

Proof. We will show that p cannot attain its maximum on $\partial\Omega$ unless it is attained at a critical point of u which is on Γ_2.

Suppose that p takes its maximum at a point $M \in \Gamma_1$. Then M cant be a critical point of u. Since $u = 0$ on Γ_1 , we have$|\nabla u| = \left|\frac{\partial u}{\partial n}\right|$and

$$\frac{\partial p}{\partial n} = 2u_n u_{nn} + 2fu_n, \tag{2.1}$$

Where u_n denotes the outward normal derivative.By introducing normal coordinates in the neighbourhood of the boundary , we can write

$$\Delta u = u_{nn} + ku_n = -f, \tag{2.2}$$

where k denotes the curvature of the boundary.Thus it follows that

$$\frac{\partial p}{\partial n} = -2ku_n^2, \tag{2.3}$$

and since Ω is convex , $\frac{\partial p}{\partial n} \leqslant 0$ at M. This contradicts Hopf's second maximum principle [5].

We now suppose that p takes its maximum at $M \in \Gamma_2$ and that M is not a critical point of u. Since $\frac{\partial u}{\partial n} = 0$ on Γ_2 , we have $|\nabla u| = \left|\frac{\partial u}{\partial t}\right|$ and

$$\frac{\partial p}{\partial n} = 2u_t u_{tn}, \tag{2.4}$$

where u_t denotes the tangential derivative of u. In terms of normal coordinates in the neighbourhood of the boundary , we have

$$u_{tn} = u_{nt} - ku_t, \tag{2.5}$$

so that

$$\frac{\partial p}{\partial n} = -2ku_t^2 \quad on\ \Gamma_2.$$

Thus we again have a contradiction of the second maximum principle when Ω is convex. The lemma, and our calculations , gives the theorem.

Example. *Let* $u \in C^3(\overline{\Omega})$ *be a positive solution of the problem*

$$\Delta u + 4u - (x_1^2 + x_2^2)\exp(\alpha^2 - x_1^2 - x_2^2) = 0 \qquad in\Omega,$$

$$u = 0 \qquad on\ \ \partial\Omega,$$

where

$$\Omega = \{x = (x_1, x_2) \backslash |x| < \alpha\}$$

and

$$f(x, u) = 4u - (x_1^2 + x_2^2)\exp(\alpha^2 - x_1^2 - x_2^2),$$

it follows from the theorem (2.1) that

$$|\nabla u|^2 + 2\int_0^u 4s - (x_1^2 + x_2^2)\exp(\alpha^2 - x_1^2 - x_2^2)ds - 2H(x, u)$$

$$\leqslant \max_{\Omega\cup\ \partial\Omega}\left[2\int_0^u 4s - (x_1^2 + x_2^2)\exp(\alpha^2 - x_1^2 - x_2^2)ds - 2H(x, u)\quad\right]$$

or

$$|\nabla u|^2 \leqslant \max_{\Omega\cup\ \partial\Omega}\left[4u^2 - 2(x_1^2 + x_2^2)\exp(\alpha^2 - x_1^2 - x_2^2)u\right]$$

$$-\left[4u^2 - 2(x_1^2 + x_2^2)\exp(\alpha^2 - x_1^2 - x_2^2)u\right]$$

.

From the above inequality , we get

$$|\nabla u|^2 \leqslant 4(u_M^2 - u^2) + 2\alpha^2 e^{\alpha^2} u_M,$$

where u_M *is the maximum of* u *in* $\Omega \cup \partial\Omega$.

3. Concluding Remarks

1. One can prove the result of the lemma for the function $p = g(u)\,|\nabla u|^2 + 2\int_0^u f(x, s)g(s)ds - 2H(x, u)$ with suitable assumptions on $g(u)$ as in [5].

2. Theorem 2.1 is also valid for $n > 2$, [5].

3.One may give an extention of the maximum principle for a uniformly elliptic equation $Lu + f(x, u) = 0$ under suitable assumptions, [5].

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