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Quasi-Complete Primary Components in Modular Abelian Group Rings over Special Rings

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ABSTRACT. Let G be a multiplicatively written p-separable abelian group and R a commutative unitary ring of prime characteristic p so that R^{p^i} has nilpotent elements for each positive integer $i \geqslant 1$. Then, we prove that, the normed unit p-subgroup $S(RG)$ of the group ring RG is quasi-complete if and only if G is a bounded p-group. This strengthens our recent results in (Internat. J. Math. Analysis, 2006) and (Scientia Ser. A - Math., 2006).

I. Introduction

Suppose that RG is the group ring of an abelian group G, written via multiplicative record as is the custom when regarding group algebras, over a commutative ring R with identity of prime characteristic p. Traditionally, $S(RG)$ denotes the Sylow group in RG consisting of all normalized p-elements of torsion of RG.

The purpose of the present paper is to find a suitable criterion when $S(RG)$ is quasi-complete; note that the class of quasi-complete primary abelian groups is defined in all details in [9] and [8] and it properly contains the class of torsion-complete ones.

The query for quasi-completeness of $S(RG)$ was started in [2] and developed in [3], [4] and [6], respectively. Paralleling, we have explored the query for torsion-completeness in [1] and [7].

Moreover, in [5] we have given another simple proof that $S(RG)$ is quasi-complete if and only if G is bounded, provided G is p-primary. Actually, we have proved the more general version of such a result for reduced thick p-groups; it is worth noting that the quasi-complete groups are obviously thick.

For the sake of completeness we briefly quote below the best known principle results at present pertaining to both torsion-completeness and quasi-completeness.

Theorem ([6]). *Suppose G is an abelian group and R is a commutative ring with 1 of prime characteristic p so that $\cap_{i<\omega} R^{p^i}$ has nilpotent elements. Then $S(RG)$ is quasi-complete if and only if G is a bounded p-group.*

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Theorem ([7]). *Let G be an abelian group with $\cap_{i<\omega} G^{p^i} = 1$ and R a commutative ring with 1 of prime characteristic p such that R^{p^i} possesses nilpotents for each $i \in \mathbb{N}$. Then $S(RG)$ is torsion-complete if and only if G is a bounded p-group.*

We here shall generalize both the statements alluded to above for quasi-complete groups by using the same ideas as in [6] and [7].

All our used notions and notation are standard and follow essentially those from [8].

II. Main Result

We recall that the group G is *p-separable* provided that $\cap_{i<\omega} G^{p^i} = 1$.

We are now in position to proceed by proving the following.

Theorem. *Let G be a p-separable abelian group and let R be a commutative unital ring of prime characteristic p such that R^{p^i} has nilpotents for all non-negative integers i. Then $S(RG)$ is quasi-complete if and only if G is a bounded p-group.*

Proof. Concerning the sufficiency, G being a bounded p-group trivially implies that so does $S(RG)$, hence quasi-complete.

As for the necessity, suppose $1 \neq S(RG)$ is quasi-complete. Note that $S(RG) = 1$ ensures that $G = 1$ (see, for example, [1]). Since, for each $q \neq p$, $G_q \subseteq \cap_{i<\omega} G^{p^i} = 1$ being p-divisible, we infer that G must be p-mixed. So, if we prove that G is torsion, hence it is of necessity p-torsion, we may apply [3] to get the claim and thereby we are finished.

And so, in order to argue this, we assume in a way of contradiction that G is torsion-free, that is $G_p = 1$. Select a proper pure and nice subgroup H of G and an element $a \in G \setminus H$. Clearly, for every integer $i \geqslant 1$, $a^{p^i} \notin G^{p^k} H$ for some natural number k, whence for all but a finite number of integers k. Otherwise, $a^{p^i} \in \cap_{k<\omega}(G^{p^k} H) = (\cap_{k<\omega} G^{p^k})H = H$. Now, because of the purity of H in G, we write $a^{p^i} \in H \cap G^{p^i} = H^{p^i}$. Whence $a \in H$, since $G_p = 1$, which is false. Even more generally, $\forall\ d \in \mathbb{N}$, $a^d \notin G^{p^k} H$ for almost all $k \in \mathbb{N}$.

It is straightforward to see that $S(RH)$ is unbounded and pure in $S(RG)$ (see, for instance, [6]). Consequently, [9] applies to show that

$$S(RG)/S(RH) \cong [D.S(RH)/S(RH)] \times [S(RG)/D.S(RH)],$$

for some $D \leqslant S(RG)$ where the first factor is divisible, hence $D \subseteq D^{p^k}.S(RH)$ for each $k \geqslant 1$, whereas the second one is torsion-complete.

If we presume that the latter direct factor $S(RG)/D.S(RH)$ is bounded, then there is a natural j with the property that $S^{p^j}(RG) \subseteq D.S(RH)$, whence $S^{p^j}(RG) = S(R^{p^j} G^{p^j}) \subseteq D^{p^k}.S(RH)$ over every $k \geqslant 1$. Thus, we consider the element $1 + r_j^{p^j}(1 - a^{p^j}) \in S(R^{p^j} G^{p^j})$ where $r_j \in R$ so that $r_j^{p^j} \neq 0$ while $r_j^{p^{j+1}} = 0$. That is why, $1 + r_j^{p^j} - r_j^{p^j} a^{p^j} \in S(R^{p^k} G^{p^k})S(RH)$ and thereby $a^{p^j} \in G^{p^k} H$ which is impossible as already observed above. Finally, we infer that $S(RG)/D.S(RH)$ is, in fact, unbounded torsion-complete.

Further, we define an infinite sequence ϕ_n of elements of $S(RG)$ in the same guise as [7], namely:

$$\phi_n = \prod_{i=1}^{n}(1 + r_i^{p^i}(1 - a^{p^i})),$$

where $r_i \in R$ such that $r_i^{p^{i+1}} = 0$ but $r_i^{p^i} \neq 0$.

Imitating [7], we write ϕ_n in a canonical record as follows:

$$\phi_n = \beta_0^{(n)} + \beta_1^{(n)} a^{u_1} + \cdots + \beta_{s_n}^{(n)} a^{u_n},$$

where $\beta_0^{(n)}, \beta_1^{(n)}, \cdots, \beta_{s_n}^{(n)} \in R$ with $\beta_0^{(n)} + \beta_1^{(n)} + \cdots + \beta_{s_n}^{(n)} = 1$, $s_n \in \mathbb{N}$ and $u_1, \cdots, u_n \in \mathbb{N}$ are sums of different powers of p.

By virtue of the arguments above, we derive $\phi_n \in S(RG) \setminus (D.S(RH))$. It is a routine technical exercise to check that ϕ_n is a Cauchy sequence, bounded by p.

Next, we define the sequence $\psi_n = \phi_n D.S(RH)$ in $S(RG)/D.S(RH)$ which, owing to ([6], Lemma), is obviously bounded by p Cauchy sequence. Utilizing the topological criterion of Kulikov for torsion-completeness from [10] (see too [8], v. II, p. 38, Theorem 70.7), one may write that for all $k \geqslant 1$ and $n \geqslant k$ there exists a fixed element $\psi \in S(RG)/D.S(RH)$ such that $\psi \in \psi_n(S(RG)/D.S(RH))^{p^k}$ holds.

Letting $\psi = (r_1 g_1 + \cdots + r_t g_t)D.S(RH)$, where $t \in \mathbb{N}$ is fixed, and since $D.S(RH) = D^{p^k}.S(RH)$ we obtain that

$$r_1 g_1 + \cdots + r_t g_t = (\beta_0^{(n)} + \beta_1^{(n)} a^{u_1} + \cdots + \beta_{s_n}^{(n)} a^{u_n}).(\alpha_{1n}^{(k)^{p^k}} c_{1n}^{(k)^{p^k}} + \cdots + \alpha_{m_n n}^{(k)^{p^k}} c_{m_n n}^{(k)^{p^k}}).$$
$$(f_{1n}^{(k)} h_{1n}^{(k)} + \cdots + f_{l_n n}^{(k)} h_{l_n n}^{(k)}),$$

where $r_1 g_1 + \cdots + r_t g_t \in S(RG), \alpha_{1n}^{(k)^{p^k}} c_{1n}^{(k)^{p^k}} + \cdots + \alpha_{m_n n}^{(k)^{p^k}} c_{m_n n}^{(k)^{p^k}} \in S(R^{p^k} G^{p^k})$ and $f_{1n}^{(k)} h_{1n}^{(k)} + \cdots + f_{l_n n}^{(k)} h_{l_n n}^{(k)} \in S(RH)$ are written in canonical form.

Because $u_n > n \geqslant k$ and as already argued above $a^{u_1} \notin G^{p^k}H, \cdots, a^{u_n} \notin G^{p^k}H$ for almost all $k \geqslant 1$, whence for some sufficiently large $k > t$, we easily see that in the left hand-side of the last equality the number of group elements in the support is precisely t in contrast to the right hand-side where this number is strictly greater than t. But this is a contradiction which unambiguously shows that $1 \neq S(RG)$ being quasi-complete guarantees that $G_p \neq 1$. Now, if we assume that G_p is unbounded, with the aid of its purity in $S(RG)$ and [9] we deduce that $S(RG)/G_p$ should be torsion-complete. Thus, according to the proof of Theorem 2 from [3] we obtain that G_p has to be bounded, against our assumption. This leads us to the fact that G_p is, ever, bounded. Knowing this, we employ the classical theorem due to Prüfer-Kulikov [10] (e.g. [8], v. I, p. 140, Chapter V, Paragraph 27, Theorem 27.5) to write that $G = G_p \times M$ for some subgroup M of G. Furthermore, $S(RM) \cong S(R(G/G_p))$ being a direct factor of $S(RG)$ is also quasi-complete. That is why, the previous step works and allows us to conclude that this is possible only when $G/G_p = 1$, i.e. when $G = G_p$. Finally, we infer that G is p-primary bounded and this finishes the proof. $\diamondsuit$

As an immediate consequence, we yield the following.

Main Theorem. *Suppose G is a p-separable abelian group and R is a commutative unital ring of prime characteristic p. Then $S(RG)$ is quasi-complete if and only if* *either R^{p^j} is without nilpotents for some $j \in \mathbb{N}$ and G_p is bounded, or R^{p^i} is with nilpotents for every $i \in \mathbb{N}$ and G is a bounded p-group.*

Proof. The first implication follows directly from [3], whereas the latter one follows from the preceding Theorem. $\diamondsuit$

In closing, we pose the following.

Problem. Assume that G is an abelian group and R is a commutative unitary ring of prime characteristic p so that R^{p^i} has nilpotent elements for any natural number $i \geqslant 1$. Then does it follow that $S(RG)$ being quasi-complete will imply that G is p-separable?

If this problem can be resolved in a positive way (we conjecture that it is so), with the foregoing Theorem at hand, the question for quasi-completeness of $S(RG)$ is completely exhausted.

Remark: In [5] all our groups in the main theorems are reduced.

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