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On the evaluation of the integral $\int_0^\infty x^{-a}(1 - \sin^b x/x^b)\, dx$

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Abstract. In the present paper we provide a closed-form evaluation of the integral $\int_0^\infty x^{-a}\left(1 - \frac{\sin^b x}{x^b}\right) dx$, where $a \in (0, 3)$ and $b \in \mathbb{N}_0$

1. Introduction

In the study of the rate of transfer of heat or mass from a force-free coupled-free particle immersed in a fluid whose velocity far from the particle is steady and varies linearly with position, the reader will find the integral [**1**]:

$$\int_0^\infty x^{-3/2}\left(1 - \frac{\sin^2 x}{x^2}\right) dx. \tag{1.1}$$

Conducting a literature search we have not found an exact expression for the generalized integral

$$I(a, b) := \int_0^\infty x^{-a}\left(1 - \frac{\sin^b x}{x^b}\right) dx. \tag{1.2}$$

Only a special case of this integral appears on the table of integrals by Gradshteyn and Rhyzik [**2**], but this case is not relevant for the physical situation described above.

The goal of this paper is to evaluate $I(a, b)$ for $b \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The expression is given in terms of the classical gamma function

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1}\, dt. \tag{1.3}$$

Theorem 1.1. Let $a \in (0, 3)$ and $b \in \mathbb{N}_0$. Then

$$I(a, b) = \frac{\pi \sec(\pi a/2)}{2^b\, \Gamma(a + b)} \sum_{k=0}^{\lfloor (b-1)/2 \rfloor} (-1)^{k+1} \binom{b}{k} (b - 2k)^{a+b-1}. \tag{1.4}$$

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2. Convergence of the integral $I(a,b)$

In this section we determine conditions on the parameters a and b that guarantee the convergence of the integral $I(a,b)$. The singularities at the origin and infinity are treated separately. Choose a point $x_0 > 0$ and write

$$I(a,b) = \int_0^{x_0} x^{-a}\left(1-\frac{\sin^b x}{x^b}\right)dx + \int_{x_0}^\infty x^{-a}\left(1-\frac{\sin^b x}{x^b}\right)dx. \tag{2.1}$$

Using a criterion of Cauchy for the convergence of integrals, it is easy to check the following result.

Lemma 2.1. The integral $I(a,b)$ converges for $1 < a < 3$, independently of b.

3. The proof of Theorem 1.1

The proof of Theorem 1.1 employs the value of a binomial sum.

Lemma 3.1. Let $m,\, p \in \mathbb{N}$. Then

$$\sum_{k=0}^m (-1)^k \binom{m}{k}(m-2k)^p = \begin{cases} 0 & \text{for } 0 \leqslant p \leqslant m-1, \\ 2^m m! & \text{for } p = m. \end{cases} \tag{3.1}$$

PROOF. Write the sum as

$$\begin{aligned} S := \sum_{k=0}^m (-1)^k \binom{m}{k}(m-2k)^p &= \sum_{k=0}^m (-1)^k \binom{m}{k} \sum_{l=1}^p \binom{p}{l}(-1)^l 2^l m^{p-l} k^l \\ &= \sum_{l=1}^p (-1)^l \binom{p}{l} 2^l m^{p-l} \sum_{k=0}^m (-1)^k \binom{m}{k} k^l. \end{aligned}$$

Now employ the standard identity

$$\sum_{k=0}^m (-1)^k \binom{m}{k} k^p = \begin{cases} 0 & \text{for } 0 \leqslant p \leqslant m-1, \\ (-1)^m m! & \text{for } p = m, \end{cases} \tag{3.2}$$

to complete the proof. □

The evaluation of $I(a.b)$ begins with

$$I(a,b) = \int_0^\infty x^{-(a+b)}\left(x^b - \sin^b x\right)dx. \tag{3.3}$$

Now use

$$x^{-(a+b)} = \frac{1}{\Gamma(a+b)}\int_0^\infty e^{-xy} y^{a+b-1}\,dy, \tag{3.4}$$

to produce

$$I(a,b) = \frac{1}{\Gamma(a+b)}\int_0^\infty \int_0^\infty e^{-xy} y^{a+b-1}\left(x^b - \sin^b y\right)dy\,dx \tag{3.5}$$

that we write as

$$I(a,b) = \frac{1}{\Gamma(a+b)} \int_0^\infty y^{a+b-1} G(y)\, dy, \tag{3.6}$$

where

$$G(y) := \int_0^\infty e^{-xy} \left(x^b - \sin^b x\right)\, dx. \tag{3.7}$$

The exchange of the order of integration admits an elementary justification.

To evaluate $G(y)$ we integrate by parts $b+1$ times. This yields, with $f(x) = x^b - \sin^b x$,

$$G(y) = \frac{1}{y^{b+1}} \int_0^\infty e^{-xy} f^{(b+1)}(x)\, dx + \sum_{k=1}^{b+1} \frac{1}{y^k} f^{(k-1)}(0). \tag{3.8}$$

Expanding f near $x=0$, we see that $f^{(k)}(0) = 0$ for $k \leqslant b$. We now use $\sin x = (e^{ix} - e^{-ix})/2$ to conclude that

$$G(y) = -\frac{i}{2^b y^{b+1}} \sum_{k=0}^{b} a_k b_k^{b+1} \frac{y + ib_k}{y^2 + b_k^2}, \tag{3.9}$$

where $a_k = (-1)^k \binom{b}{k}$ and $b_k = b - 2k$.

Lemma 3.2. The function G can be expressed as

$$G(y) = \frac{1}{2^{b-1} y^{b+1}} \sum_{k=0}^{\lfloor (b-1)/2 \rfloor} (-1)^k \binom{b}{k} \frac{(b-2k)^{b+2}}{y^2 + (b-2k)^2}. \tag{3.10}$$

Proof. This follows directly by using Lemma 3.1 and reducing (3.9) by distinguishing cases according to parity. □

The expression for G now yields

$$I(a,b) = \frac{1}{2^b \Gamma(a+b)} \int_0^\infty \frac{v^{(a-3)/2}}{v+1}\, dv \times \sum_{k=0}^{\lfloor (b-1)/2 \rfloor} (-1)^k \binom{b}{k} \frac{(b-2k)^{b+2}}{y^2 + (b-2k)^2}. \tag{3.11}$$

The final step in the proof of Theorem 1.1 is to identify the integral in terms of Euler's beta function via

$$\int_0^\infty \frac{v^{x-1}\, dv}{1+v} = B(x, 1-x) = \frac{\pi}{\sin \pi x}, \tag{3.12}$$

and use $x = (a-1)/2$. The proof of Theorem 1.1 is complete.

4. Special cases

In this section we illustrate Theorem 1.1 in some particular examples.

Example 4.1. The table of integrals of Gradshteyn and Rhyzik [**2**] contains the particular case $a = 2$ and $b \in \mathbb{N}_0$. Our expression yields

$$I(2,b) = \frac{\pi}{2^b\,(b+1)!} \sum_{k=0}^{\lfloor (b-1)/2 \rfloor} (-1)^k \binom{b}{k} (b-2k)^{b+1}. \tag{4.1}$$

This appears as 3.829.1 on page 459 of [**2**].

Example 4.2. The main formula in Theorem 1.1 yields

$$I(a,1) = -\frac{\pi\,\sec(\pi a/2)}{2\Gamma(1+a)}. \tag{4.2}$$

In particular, for $a = \frac{3}{2}$ we obtain

$$\int_0^\infty x^{-3/2}\left(1 - \frac{\sin x}{x}\right)dx = \frac{2\sqrt{2\pi}}{3}. \tag{4.3}$$

Example 4.3. The main formula in Theorem 1.1 yields

$$I(a,2) = -\frac{\pi\,2^{a-1}\,\sec(\pi a/2)}{\Gamma(2+a)}. \tag{4.4}$$

In particular, for $a = \frac{3}{2}$ we obtain

$$\int_0^\infty x^{-3/2}\left(1 - \frac{\sin^2 x}{x^2}\right)dx = \frac{16\sqrt{\pi}}{15}. \tag{4.5}$$

Example 4.4. The case $b = 3$ in Theorem 1.1 now yields

$$I(a,3) = \frac{(3-3^{2+a})\,\pi\,\sec(\pi a/2)}{8\Gamma(3+a)}. \tag{4.6}$$

In particular, for $a = \frac{3}{2}$ we obtain

$$\int_0^\infty x^{-3/2}\left(1 - \frac{\sin^3 x}{x^3}\right)dx = \frac{2}{35}(9\sqrt{3}-1)\sqrt{2\pi}. \tag{4.7}$$

References

[1] G. K. Batchelor, Mass transfer from a particule suspended in fluid with a steady linear ambiant velocity. J. Fluid Mech., 95(2), 369-400, 1979.

[2] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, London, 2000.

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