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The Multivariable Voigt Functions and their representations

M.A.Pathan

Abstract. In the present paper it will be shown that generalized Voigt function is expressible in terms of a combination of Kampé de Fériet's functions. We also give further generalizations (involving multivariables) of Voigt functions in terms of series and integrals which are specially useful when the parameters take on special values. The results of multivariable Hermite polynomials are used with a view to obtaining explicit representations of generalized Voigt functions.

The Voigt functions $K(x,y)$ and $L(x,y)$ occur frequently in a wide variety of problems in astrophysical spectroscopy, emission, absorption and transfer of radiation in heated atmosphere, plasma dispersion, neutron reactions and indeed in the several diverse field of physics. Following the work of Srivastava and Miller [12] closely, Klusch [7] has given a generalization of the Voigt functions in the form

$$\Omega_{\mu,\nu}(x,y,z) = \sqrt{\frac{x}{2}}\int_0^\infty t^\mu e^{-yt-zt^2}\, J_\nu(xt)\,dt \tag{1.1}$$

$$= \frac{z^{-\alpha}x^{\nu+\frac{1}{2}}}{2^{\nu+\frac{1}{2}}\Gamma(\nu+1)}\left\{\Gamma(\alpha)\psi_2\left[a;\,\nu+1,\frac{1}{2};\,-\frac{x^2}{4z},-\frac{y^2}{4z}\right]\right.$$

$$\left. -\frac{y}{\sqrt{z}}\Gamma(\alpha+\frac{1}{2})\,\psi_2\left[\alpha+\frac{1}{2};\,\nu+1,\frac{3}{2};\,-\frac{x^2}{4z},-\frac{y^2}{4z}\right]\right\} \tag{1.2}$$

$$(\alpha=(\mu+\nu+1)/2,\ x,y,z\in\mathbb{R}^+,\ R(\mu+\nu)>-1)$$

where ψ_2 denotes one of Humbert's confluent hypergeometric function of two variables, defined by [11; p.59]

$$\psi_2[\alpha;\,\gamma,\gamma';\,x,y] = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m\,(\gamma')_n}\,\frac{x^m\,y^n}{m!\,n!},\ \max\{|x|,|y|\}<\infty \tag{1.3}$$

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and the classical Bessel function $J_\nu(x)$ is defined by [11]

$$J_\nu(x) \;=\; \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)},\;(z\in\mathbb{C}\setminus(-\infty,0)) \tag{1.4}$$

so that

$$K(x,y) \;=\; \Omega_{1/2,-1/2}\left(x,y,\frac{1}{4}\right) \text{ and } L(x,y) \;=\; \Omega_{1/2,1/2}\left(x,y,\frac{1}{4}\right) \tag{1.5}$$

The present work is inspired by the frequent requirements of various properties of Voigt functions in the analysis of certain applied problems. Our aim is to further introduce two more generalizations of (1.1) and another interesting explicit representation of (1.1) in terms of Kampé de Feriét series $F^{p:q;r}_{l:m;n}$ [see (11, p.63)]

In an attempt to generalize (1.1), we first investigate here the generalized Voigt function $\Omega^{(j)}_{\mu,\nu}(x,y,z)$ defined by

$$\Omega^{(j)}_{\mu,\nu} \;=\; \Omega^{(j)}_{\mu,\nu}(x,y,z) \;=\; \sqrt{\frac{x}{2}}\int_0^\infty t^\mu e^{-yt-zt^j}\,J_\nu(xt)\,dt \tag{1.6}$$

where j is a positive integer, $x,y,z\in\mathbb{R}^+$ and $\mathrm{Re}(\mu+\nu)>-1$. Clearly, we have

$$\Omega^{(2)}_{\mu,\nu}(x,y,z) \;=\; \Omega_{\mu,\nu}(x,y,z) \tag{1.7}$$

and $\Omega^{(1)}_{\mu,\nu}(x,y,z)$ is the classical Laplace transform of $t^\mu J_\nu(xt)$. Furthermore

$$\Omega^{(2)}_{1/2,-1/2}\left(x,y,\frac{1}{4}\right) \;=\; K(x,y) \text{ and } \Omega^{(2)}_{1/2,-1/2}\left(x,y,\frac{1}{4}\right) \;=\; L(x,y)$$

Relationships (1.5) and (1.7) can indeed be used to obtain another interesting generalization (and unification) of the results (1.1) and (1.6). The multivariable extension of the Voigt function is defined by

$$\Lambda_{\mu,\nu} = \Lambda_{\mu,\nu}(x,y,z,x_1,\cdots,x_m) = \sqrt{\frac{x}{2}}\int_0^\infty t^\mu\, e^{-yt-zt^2+\sum\limits_{j=1}^{m}x_jt^j}\,J_\nu(xt)\,dt \tag{1.8}$$

where m is a positive integer, $x,y,z,\ x_1,\cdots,x_m\in\mathbb{R}^+$ and $\mathrm{Re}(\mu+\nu)>-1$.

Upon comparing the definitions (1.4) and (1.6) with (1.8), we readily obtain the relationships

$$\Lambda_{\mu,\nu}(x,y,0,0,\cdots,0,-z) \;=\; \Lambda_{\mu,\nu}(x,y,z,0,\cdots,0) \;=\; \Omega^{(j)}_{\mu,\nu}(x,y,z)$$
$$\Lambda_{\mu,\nu}(x,y,z,x_1) \;=\; \Omega_{\mu,\nu}(x,y-x_1,z),$$
$$\Lambda_{\mu,\nu}(x,y,z,x_1,x_2) \;=\; \Omega_{\mu,\nu}(x,y-x_1,z-x_2).$$

We also record here

$$\lim_{x\to0}\Lambda_{\mu,\nu}(x,y,z,x_1,\cdots,x_m) \;=\; \frac{1}{\Gamma(\nu+1)}\int_0^\infty t^{\mu+\nu}\,e^{-yt-zt^2+\sum\limits_{j=1}^{m}x_jt^j}\,dt \tag{1.9}$$

2. Preliminary Results

Apart from the well-known embedding of Hermite polynomials into the confluent hypergeometric functions [5,6] and the extension of multivariable Hermite polynomials [1,4] and to Hermite 2D polynomials [14], some other specific generalizations were considered in literature. One such generalization are the Gould-Hopper generalized Hermite polynomials

$$H_n^{(m)}(z,y) \;=\; n! \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{y^k\, z^{n-mk}}{k!\,(n-mk)!},\;\; m=1,2,\cdots \tag{2.1}$$

which for $m=1$ contain the binomial expansion and for $m=2$ the usual Hermite polynomials in the form of Appell and Kampé de Feriét [11].

The multivariable Hermite polynomials $H_n^{(m)}(\{x\}_1^m)$ are specified by the generating function [4]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, H_n^{(m)}(\{x\}_1^m) \;=\; e^{\sum\limits_{j=1}^{m} x_j t^j} \tag{2.2}$$

where $\{x\}_1^m = x_1, x_2, \cdots, x_m$.

It is now convenient to specialize (2.2) by writing

$$H_n^{(1)}(x) \;=\; H_n^{(2)}(2x,-1) \;=\; H_n(x) \tag{2.3}$$

where $H_n(x_1,x_2)$ are two variable Hermite-Kampé de Feriét polynomials [2] defined by the series

$$H_n(x_1,x_2) \;=\; n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(x_2)^k\,(x_1)^{n-2k}}{k!\,(n-2k)!} \tag{2.4}$$

and $H_n(x)$ are Hermite polynomials [11].

$$H_n^{(3)}(2x,-y,z) \;=\; H_n(x,y,z) \tag{2.5}$$

where $H_n(x,y,z)$ are three variable generalized Hermite polynomials defined by the generating function [see (1, p.511)]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, H_n(x,y,z) \;=\; e^{2xt-yt^2+zt^3} \tag{2.6}$$

Furthermore, an obvious specialization of (2.2) is given by Dattoli *et al.* [3] in the form

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, h_n(x,y;\xi) \;=\; e^{2xt-t^2+2y\xi t-\xi^2 t^2} \tag{2.7}$$

straightforwardly yielding the following expansion in terms of ordinary Hermite polynomials

$$H_n^{(2)}(2x+2y\xi,-\xi^2-1) \;=\; h_n(x,y;\xi) \;=\; \sum_{s=0}^{n} \binom{n}{s} \xi^s\, H_{n-s}(x)\, H_s(y) \tag{2.8}$$

which can also be written as

$$h_n(x,y;\xi) \;=\; (1+\xi^2)^{n/2}\, H_n\left(\frac{x+\xi y}{\sqrt{1+\xi^2}}\right) \tag{2.9}$$

3. Another Representation for $\Omega_{\mu,\nu}(x,y,z)$

Formula (1.1) happens to be a special case of the result [13, p.53(1.24) (see also [8])] which we proved, a couple of years ago, in an attempt to provide an explicit expression of the generalized Voigt function associated with Humbert confluent hypergeometric function $\psi_2^{(2)}$ of two variables. Now we make use of the result [10, p.101(2)]

$$\int_0^\infty t^\alpha\, e^{-zt^2}\, J_\nu(xt)\, dt = \frac{\sin\nu\pi\,\Gamma(\alpha+1)/2}{2\nu\pi\, z^{(\alpha+1)/2}}\, {}_2F_2\left[1, \frac{\alpha+1}{2};\, 1+\frac{\nu}{2}, 1-\frac{\nu}{2};\, -\frac{x^2}{4z}\right]$$
$$-\frac{x\sin\nu\pi\,\Gamma(\alpha+2)/2}{2\pi(1-\nu^2)\, z^{(\alpha+2)/2}}\, {}_2F_2\left[1, \frac{\alpha+2}{2};\, \frac{3+\nu}{2}, \frac{3-\nu}{2};\, -\frac{x^2}{4z}\right] \tag{3.1}$$
$$[\operatorname{Re} z,\ \operatorname{Re}(\alpha+1)>0]$$

to derive another class of representations of Voigt function $\Omega_{\mu,\nu}$ associated with Kampé de Feriét hypergeometric functions of two variables $F^{p:q;r}_{l:m;n}$ (see, e.g., [11, p.63]) in the following form

$$\Omega_{\mu,\nu}(x,y,z) = \frac{x^{1/2}\sin\nu\pi}{2\sqrt{2}\pi\nu\, z^{(\alpha+1)/2}} \left\{ \frac{\Gamma(\alpha+1)}{2}\, F^{1:0;1}_{0:1;2}\left[\begin{matrix} \frac{\alpha+1}{2} : \_;\ 1 & ; \\ & \frac{y^2}{4z}, -\frac{x^2}{4z} \\ \_ : \frac{1}{2}; 1+\frac{\nu}{2}, 1-\frac{\nu}{2} & ; \end{matrix}\right]\right.$$
$$-\frac{\nu\, x\,\Gamma(\alpha+2)/2}{z^{1/2}(1-\nu^2)}\, F^{1:0;1}_{0:1;2}\left[\begin{matrix} \frac{\alpha+2}{2} : \_;\ 1 & ; \\ & \frac{y^2}{4z}, -\frac{x^2}{4z} \\ \_ : \frac{1}{2}; \frac{3+\nu}{2}, \frac{3-\nu}{2} & ; \end{matrix}\right] - \frac{\nu\, y\Gamma(\alpha+2)/2}{z^{1/2}}$$
$$\times F^{1:0;1}_{0:1;2}\left[\begin{matrix} \frac{\alpha+2}{2} : \_;\ 1 & ; \\ & \frac{y^2}{4z}, -\frac{x^2}{4z} \\ \_ : \frac{3}{2}; 1+\frac{\nu}{2}, 1-\frac{\nu}{2} & ; \end{matrix}\right] + \frac{\nu\, xy\Gamma(\alpha+3)/2}{z(1-\nu^2)}$$
$$\times F^{1:0;1}_{0:1;2}\left[\begin{matrix} \frac{\alpha+3}{2} : \_;\ 1 & ; \\ & \frac{y^2}{4z}, -\frac{x^2}{4z} \\ \_ : \frac{3}{2}; \frac{3+\nu}{2}, \frac{3-\nu}{2} & ; \end{matrix}\right] \quad [\operatorname{Re}(\alpha+\nu+1)>0,\ \operatorname{Re} z>0] \tag{3.2}$$

To prove (3.2), we expand the exponential function e^{-yt} in (1.1) and then integrate the resulting (absolutely convergent) series term by term with the help of the result (3.1). Now using series expansion of ${}_2F_2$, we have

$$\Omega_{\mu,\nu}(x,y,z) = \sqrt{\frac{x}{2}}\sum_{r=0}^{\infty}\frac{(-y)^r}{r!}\left[\frac{\sin\nu\pi\,\Gamma(\alpha+1+r)/2}{2\nu\pi\,z^{(\alpha+1+r)/2}}\sum_{s=0}^{\infty}\frac{(1)_s\left(\frac{\alpha+r+1}{2}\right)_s}{\left(1+\frac{\nu}{2}\right)_s\left(1-\frac{\nu}{2}\right)_s\,s!}\left(\frac{-x^2}{4z}\right)^s\right.$$

$$\left.-\frac{x\sin\nu\pi}{2\pi(1-\nu^2)}\frac{\Gamma(\alpha+r+2)/2}{z^{(\alpha+r+2)/2}}\sum_{s=0}^{\infty}\frac{(1)_s\left(\frac{\alpha+r+2}{2}\right)_s}{\left(\frac{3+\nu}{2}\right)_s\left(\frac{3-\nu}{2}\right)_s\,s!}\left(\frac{-x^2}{4z}\right)^s\right] \tag{3.3}$$

Separating the r-series into its even and odd terms (see [11, p.200, Equation 3.1(1)]) and using $(a+r)_s = (a)_{r+s}/(a)_r$, we get (3.2).

4. Explicit Representation for $\Omega^{(j)}_{\mu,\nu}$

To obtain the various explicit representations for the generalized Voigt function $\Omega^{(j)}_{\mu,\nu}$, we first make use of the series representation (1.4) in (1.6) and integrate the resulting series term by term with the help of the result

$$\int_0^\infty t^\mu e^{-pt-\beta t^\lambda}\,dt = \sum_{r=0}^{\infty}\frac{(-\beta)^r\,\Gamma(\mu+1+\lambda r)}{r!\,p^{\mu+1+\lambda r}}, \tag{4.1}$$

$$(\mathrm{Re}(\mu+1)>0,\ \mathrm{Re}\,p>0 \text{ and } \lambda>0)$$

We thus obtain

$$\Omega^{(j)}_{\mu,\nu}(x,y,z) = \frac{x^{\nu+\frac{1}{2}}}{2^{\nu+\frac{1}{2}}\,y^{\mu+\nu+1}}\sum_{r,m=0}^{\infty}\frac{\Gamma(\mu+\nu+2m+jr+1)}{\Gamma(\nu+m+1)\,m!\,r!}\left(\frac{-z}{y^j}\right)^r\left(\frac{-x^2}{4y^2}\right)^m \tag{4.2}$$

Setting $j=2$ and then using Legendre's duplication formula [11, p.23(26)], (4.2) would reduce to the following explicit expansions of the generalized Voigt function

$$\Omega^{(2)}_{\mu,\nu}(x,y,z) = \Omega_{\mu,\nu}(x,y,z) = \frac{x^{\nu+\frac{1}{2}}\,\Gamma(\mu+\nu+1)}{2^{\nu+\frac{1}{2}}\,y^{\mu+\nu+1}\,\Gamma(\nu+1)}$$

$$\times F^{2:0;0}_{0:1;0}\left[\begin{matrix}\frac{\mu+\nu+1}{2},\frac{\mu+\nu+2}{2} & :\,\_;\,\_ & ; \\ & & \frac{-x^2}{y^2},\frac{-4z}{y^2} \\ \_ & :\,\nu+1;\,\_ & ;\end{matrix}\right] \tag{4.3}$$

which is given by recently, though in a slightly specialized form (for $z=\frac{1}{4}$), by Pathan and Shahwan [9].

Comparison of Voigt functions $\Omega_{\mu,\nu}(x,y,z)$ given by (1.2), (3.2) and (4.3) expressed in three different forms would yield the transformations of Humbert's function $\psi_2^{(2)}$ and Kampé de Feriét function $F^{2:0;0}_{0:1;0}$ in terms of Kampé de Feriét function $F^{2:0;0}_{0:1;0}$. No simple transformation of this type seem to exist for the hypergeometric series of one variable (and, naturally also for their generalizations in two and more variables).

5. Explicit Representations for $\Lambda_{\mu,\nu}$

The use of (2.2) can be exploited to obtain the series representations of (1.8). We have indeed

$$\Lambda_{\mu,\nu}(x,y,z,x_1,\cdots,x_m) = \sqrt{\frac{x}{2}}\sum_{n=0}^{\infty}\frac{1}{n!}\,H_n^{(m)}(\{x\}_1^m)\int_0^{\infty} t^{\mu+n}\,e^{-yt-zt^2}\,J_\nu(xt)\,dt \quad (5.1)$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\,H_n^{(m)}(\{x\}_1^m)\,\Omega_{\mu+n,\nu}(x,y,z) \quad (5.2)$$

by applying (1.1) to the integral on the right of (5.1)

It may be of interest to observe here that by letting $x \to 0$ in (5.1) and using

$$\lim_{x\to 0} x^{-\nu}\,J_\nu(x) = \frac{1}{2^\nu\,\Gamma(\nu+1)}$$

and [5, 146(24)]

$$\int_0^{\infty} t^{\sigma} e^{-yt-zt^2}\,dt = 2^{(\sigma+1)/2}\,\Gamma(\sigma+1)\,e^{y^2/8z}\,D_{-\sigma-1}\left(\sqrt{\frac{y}{2z}}\right)\ (\mathrm{Re}(\sigma+1)>0,\ \mathrm{Re}\,y>0) \quad (5.3)$$

where $D_{-\nu}(x)$ is parabolic cylinder function [11], we will be able to obtain

$$\int_0^{\infty} t^{\mu}\,e^{-yt-zt^2+\sum\limits_{j=1}^{m} x_j t^s}\,dt = \sum_{n=0}^{\infty}\frac{2^{n/2}(\mu+1)_n}{n!}\,H_n^{(m)}\,(\{x\})_1^m)D_{-\mu-n-1}\left(\sqrt{\frac{y}{2z}}\right) \quad (5.4)$$

Clearly, (5.4) is equivalent to (1.9).

For $m = 1$, (5.2) gives

$$\Lambda_{\mu,\nu}(x,y,z,x_1) = \Omega_{\mu,\nu}(x,y,-x_1,z) = \sum_{n=0}^{\infty}\frac{(x_1)^n}{n!}\,\Omega_{\mu+n,\nu}(x,y,z) \quad (5.5)$$

Set $m = 2$, $x_1 = 2x$ and $x_2 = -1$ in (5.2) and use (2.3) to get

$$\Lambda_{\mu,\nu}(x,y,z,2x,-1) = \Omega_{\mu,\nu}(x,y-2x,z+1) = \sum_{n=0}^{\infty}\frac{1}{n!}\,H_n(x)\,\Omega_{\mu,\nu}(x,y,z) \quad (5.6)$$

Set $m = 3$ in (5.2) to get

$$\Lambda_{\mu,\nu}(x,y,z,x_1,x_2,x_3) = \sqrt{\frac{x}{2}}\int_0^{\infty} e^{-(y-x_1)t-(z-x_2)t^2+x_3t^3}\,J_\nu(xt)\,dt$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\,H_n^{(3)}(x_1,x_2,x_3)\Omega_{\mu,\nu}(x,y,z) \quad (5.7)$$

where $H_n^{(3)}(x_1,x_2,x_3)$ is defined by (2.5).

For $m = 2$, $x_1 = 2x + 2y\xi$ and $x_2 = -\xi - 1$, (5.2) gives

$$\sum_{n=0}^{\infty}\frac{1}{n!}\,h_n(x,y;\xi)\,\Omega_{\mu,\nu}(x,y,z) = \Omega_{\mu,\nu}(x,y-2x-2y\xi,z+\xi^2+1) \quad (5.8)$$

where $h_n(x, y; \xi)$ is defined by (2.8) (or its equivalent form (2.9)).

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Department of Mathematics,
Aligarh Muslim University,
Aligarh-202002,
India

E-mail address: mapathan@gmail.com