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A note on $\aleph_0$-spaces

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Abstract. In this paper, we prove that a space is an $\aleph_0$-space iff it is an $\aleph_1$-compact space with a σ-weakly hereditarily closure-preserving strong *cs*-network. As an application of this result, we prove that a space with a nontrivial convergent sequence is an $\aleph_0$-space iff it has a σ-weakly hereditarily closure-preserving strong *cs*-network.

In [**5**], Y. Tanaka proved a space X has a σ-hereditarily closure-preserving strong *cs*-network iff X is an $\aleph_0$-space, or a σ-closed discrete space in which each compact subset is finite [**5**, Theorem A]. Takeing the implication "hereditarily closure-preserving $\Longrightarrow$ weakly hereditarily closure-preserving" into account, an interesting work is to investigate relations between $\aleph_0$-spaces and spaces with σ-weakly hereditarily closure-preserving strong *cs*-network. In this paper, we prove that a space is an $\aleph_0$-space iff it is an $\aleph_1$-compact space with a σ-weakly hereditarily closure-preserving strong *cs*-network. As an application of this result, we prove that a space with a nontrivial convergent sequence is an $\aleph_0$-space iff it has a σ-weakly hereditarily closure-preserving strong *cs*-network.

Throughout this paper, all spaces are assumed to be regular and T_1. $\mathbb{N}$ denotes the set of all natural numbers. Let X be a space and $P \subset X$. The closure of P is denoted by $\overline{P}$. Let $\mathcal{P}$ be a family of subsets of X and $x \in X$. Then $\bigcup \mathcal{P}$ and $(\mathcal{P})_x$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$ and the subfamily $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$ respectively.

Definition 1. [**1**]. Let $\mathcal{P}$ be a family of subsets of a space X.

(1) $\mathcal{P}$ is called closure-preserving if $\overline{\bigcup \mathcal{P}'} = \bigcup\{\overline{P} : P \in \mathcal{P}'\}$ for each $\mathcal{P}' \subset \mathcal{P}$.

(2) $\mathcal{P}$ is called hereditarily closure-preserving if a family $\{H(P) : P \in \mathcal{P}\}$ is closure-preserving for each $H(P) \subset P \in \mathcal{P}$.

(3) $\mathcal{P}$ is called weakly hereditarily closure-preserving if a family $\{\{x_P\} : P \in \mathcal{P}\}$ is closure-preserving for each $x_P \in P \in \mathcal{P}$.

Obviously, each hereditarily closure-preserving family is closure-preserving and weakly hereditarily closure-preserving.

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Definition 2. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X, where $\mathcal{P}_x \subset (\mathcal{P})_x$.

(1) $\mathcal{P}$ is called a strong cs-network of X [**5**], if whenever sequence $\{x_n\}$ converges to x such that $\{x_n : n \in \mathbb{N}\} \bigcup \{x\} \subset U$ with U open in X, there is $P \in \mathcal{P}$ such that $\{x_n : n \in \mathbb{N}\} \bigcup \{x\} \subset P \subset U$.

(2) $\mathcal{P}$ is called a wcs^*-network of X [**5**], if whenever sequence $\{x_n\}$ converges to x such that $\{x_n : n \in \mathbb{N}\} \bigcup \{x\} \subset U$ with U open in X, there is $P \in \mathcal{P}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k} : k \in \mathbb{N}\} \subset P \subset U$.

(3) $\mathcal{P}$ is called a k-network of X [**4**], if whenever $K \subset U$ with K compact in X and U open in X, there is a finite $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{F} \subset U$.

Definition 3. (1) A space X is called an $\aleph_0$-space [**3**] if X has a countable k-network.

(2) A space X is called an $\aleph_1$-compact space if each closed discrete subspace of X is countable.

Remark 4. It is known that a space is an $\aleph_0$-space iff it has a countable strong cs-network iff it has a countable wcs^*-network [**5**, Proposition C].

Theorem 5. *If a space X is an $\aleph_1$-compact space with a σ-weakly hereditarily closure-preserving strong cs-network, then X is an $\aleph_0$-space.*

Proof. Let X be an $\aleph_1$-compact space with a σ-weakly hereditarily closure-preserving strong cs-network $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$, where each $\mathcal{P}_n$ is weakly hereditarily closure-preserving.

For each $n \in \mathbb{N}$, put $D_n = \{x \in X : \mathcal{P}_n$ is not point-countable at $x\}$ and put $\mathcal{P}'_n = \{P - D_n : P \in \mathcal{P}_n\}$.

Claim 1. $\mathcal{P}'_n$ is countable.

If $\mathcal{P}'_n = \{P - D_n : P \in \mathcal{P}_n\}$ is not countable, then there is an uncountable subfamily $\{P_\alpha : \alpha \in \Lambda\}$ of $\mathcal{P}_n$ such that $P_\alpha - D_n \neq \emptyset$ for each $\alpha \in \Lambda$ and $P_\alpha - D_n \neq P_{\alpha'} - D_n$ if $\alpha \neq \alpha'$, where Λ is an uncountable index set. Take $x_\alpha \in P_\alpha - D_n$ for each $\alpha \in \Lambda$. Since $\mathcal{P}_n$ is weakly hereditarily closure-preserving, $\{x_\alpha : \alpha \in \Lambda\}$ is a closed discrete subspace of X. Note that X is $\aleph_1$-compact, $\{x_\alpha : \alpha \in \Lambda\}$ is countable. So there is an uncountable subset Λ' of Λ such that $\{x_\alpha : \alpha \in \Lambda'\} = \{x\}$ for some $x \in X$. Thus $\mathcal{P}_n$ is not point-countable at x. This contradicts that $x \notin D_n$. So $\{P - D_n : P \in \mathcal{P}_n\}$ is countable.

Claim 2. D_n is a countable closed discrete subspace of X.

If D_n is not countable, then there is an uncountable subset $D'_n = \{y_\beta : \beta < \omega_1\}$ of D_n. Take $y_1 \in P_1$ for some $P_1 \in \mathcal{P}_n$. $\mathcal{P}_n$ is not point-countable at y_2, so $y_2 \in P_2$ for some $P_2 \in \mathcal{P}_n - \{P_1\}$. By transfinite induction, we can obtain a subfamily $\{P_\beta : \beta < \omega_1\}$ of $\mathcal{P}_n$ such that $P_\beta \in \mathcal{P}_n - \{P_\gamma : \gamma < \beta\}$ and $y_\beta \in P_\beta$ for each $\beta < \omega_1$. Thus $D'_n = \{y_\beta : \beta < \omega_1\}$ is an uncountable closed discrete subspace of X because $\mathcal{P}_n$ is weakly hereditarily closure-preserving. This contradicts $\aleph_1$-compactness of X. So D_n is countable. By a similar way as in the proof of that D'_n is a closed discrete subspace of X, It is easy to prove that D_n is a closed discrete subspace of X.

Put $\mathcal{U}_n = \mathcal{P}'_n \bigcup \{\{x\} : x \in D_n\}$ for each $n \in \mathbb{N}$ and put $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$. Then $\mathcal{U}$ is countable from Claim 1 and Claim 2. By Remark 4, it suffices to prove that $\mathcal{U}$ is a wcs^*-network of X.

Let $\{x_n\}$ be a sequence converging to x such that $\{x_n : n \in \mathbb{N}\} \bigcup \{x\} \subset U$ with U open in X. Since $\mathcal{P}$ is a strong cs-network of X, there is $P \in \mathcal{P}_m$ for some $m \in \mathbb{N}$ such that $\{x_n : n \in \mathbb{N}\} \bigcup \{x\} \subset P \subset U$.

Case 1. Assume $x_n = x$ for infinitely many $n \in \mathbb{N}$. Then there is a subset $\{n_k : k \in \mathbb{N}\}$ of $\mathbb{N}$ such that $x_{n_k} = x$ for each $k \in \mathbb{N}$. If $x \in D_m$, then $\{x\} \in \mathcal{U}_m \subset \mathcal{U}$ and $\{x_{n_k} : k \in \mathbb{N}\} = \{x\} \subset U$. If $x \notin D_m$, then $P - D_m \in \mathcal{U}$ and $\{x_{n_k} : k \in \mathbb{N}\} = \{x\} \subset P - D_m \subset U$.

Case 2. Assume $x_n \neq x$ for all but finitely many $n \in \mathbb{N}$. Then $S = \{x_n : n \in \mathbb{N}\} \bigcup \{x\}$ is infinite. Note that $S \bigcap D_m$ is compact in D_m, $S \bigcap D_m$ is finite, so $S \bigcap (P - D_m)$ is infinite. Thus, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k} : k \in \mathbb{N}\} \subset P - D_m$. It is clear that$P - D_m \in \mathcal{U}_m \subset \mathcal{U}$ and $P - D_m \subset U$.

By the above two case, $\mathcal{U}$ is a wcs^*-network of X. □

Note that each $\aleph_0$-space is Lindelöf and hereditarily separable, and each Lindelöf space or hereditarily separable space is $\aleph_1$-compact. We have the following corollary immediately.

COROLLARY 6. *A space X is an $\aleph_0$-space iff X has a σ-weakly hereditarily closure-preserving strong cs-network and anyone of the following conditions holds.*

(1) X is a Lindelöf space.

(2) X is a hereditarily separable space.

(3) X is an $\aleph_1$-compact space

THEOREM 7. *Let a space X have a nontrivial convergent sequence. Then X is an $\aleph_0$-space iff X has a σ-weakly hereditarily closure-preserving strong cs-network.*

PROOF. We only need to prove sufficiency. Let X has a σ-weakly hereditarily closure-preserving strong cs-network $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$, where each $\mathcal{P}_n$ is weakly hereditarily closure-preserving. By Theorem 5, it suffices to prove that X is $\aleph_1$-compact. If X is not $\aleph_1$-compact, then there is a uncountable closed discrete subspace Y. Let $\{x_n\}$ be a nontrivial sequence converging to x. Without loss of generality, assume $x_n \neq x$ for each $n \in \mathbb{N}$ and $x_n \neq x_m$ for all $n \neq m$. Put $S = \{x_n : n \in \mathbb{N}\} \bigcup \{x\}$, then $Y - S$ is uncountable. Let $Y - S = \{y_\alpha : \alpha \in \Lambda\}$, where Λ is an uncountable index set. For each $\alpha \in \Lambda$, put $U_\alpha = X - (Y - (S \bigcup \{y_\alpha\}))$, then $S \bigcup \{y_\alpha\} \subset U_\alpha$ with U_α open in X. There is $P_\alpha \in \mathcal{P}_{n_\alpha}$ for some $n_\alpha \in \mathbb{N}$ such that $S \bigcup \{y_\alpha\} \subset P_\alpha \subset U_\alpha$ because $\mathcal{P}$ is a strong cs-network of X. Thus there is an uncountable subset Λ' of Λ and $k \in \mathbb{N}$ such that $n_\alpha = k$ for all $\alpha \in \Lambda'$, that is, $P_\alpha \in \mathcal{P}_k$ for all $\alpha \in \Lambda'$. If $\alpha \neq \beta$ and $\alpha, \beta \in \Lambda'$, then $P_\alpha \neq P_\beta$ because $y_\alpha \in P_\alpha \subset X - \{y_\beta\}$ and $y_\beta \in P_\beta \subset X - \{y_\alpha\}$. So $\{P_\alpha : \alpha \in \Lambda'\} (\subset \mathcal{P}_k)$ is weakly hereditarily closure-preserving. Since Λ' is uncountable and $S \subset \bigcap\{P_\alpha : \alpha \in \Lambda'\}$, for each $n \in \mathbb{N}$, we can take $\alpha_n \in \Lambda'$ such that $x_n \in P_{\alpha_n}$ and $\alpha_{n+1} \in \Lambda' - \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Thus $\{x_n : n \in \mathbb{N}\}$ is a closed discrete subspace of X. This contradicts that $x \in \overline{\{x_n : n \in \mathbb{N}\}}$. So X is $\aleph_1$-compact. □

Recall a space X is sequential [**2**] if $U \subset X$ is open in X iff for each $x \in U$, each sequence $\{x_n\}$ converging to x, then $\{x_n : n > k\} \bigcup \{x\} \subset U$ for some $k \in \mathbb{N}$.

COROLLARY 8. *If a space X is a sequential space with a σ-weakly hereditarily closure-preserving strong cs-network, then X is an $\aleph_0$-space or X is a discrete space.*

Proof. Let X be a sequential space with a σ-weakly hereditarily closure-preserving strong cs-network. If X is not a discrete space, then there is $x \in X$ such that x is not open in X. Since X is sequential, there is nontrivial sequence converging to x. By Theorem 7, X is an $\aleph_0$-space. □

References

1. D.K.Burke, R.Engelking and D.Lutzer, Hereditarily closure-preserving and metrizability, Proc. Amer. Math. Soc., 51(1975), 483-488.
2. S.P.Franklin, Spaces in which sequence suffice, Fund. Math., 57(1965), 107-115.
3. E.A.Michael, $\aleph_0$-spaces, J. Math. Mech., 15(1966), 983-1002.
4. P.O'Meara, On paracompactness in function spaces with the compact-open topology, Proc. Amer. Math. Soc., 29(1971), 183-189.
5. Y.Tanaka, Theory of k-networks, Questions and Answers in General Topology, 12(1994), 139-164.

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