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Singularities of quasiregular mappings on Carnot groups

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Abstract. In 1970 Poletskiĭ applied the method of the module of a family of curves to describe behavior of quasiregular mappings (in another terminology mappings with bounded distortion) in $\mathbb{R}^n$. In the present paper we generalize a result by Poletskiĭ and study a singular set of a quasiregular mapping using the method of the module of a families of curves on Carnot groups.

1. Introduction

A mapping with bounded distortion is a natural generalization of an analytic function of one complex variable to the Euclidean space of the dimension $n > 2$. It was firstly introduced and studied by Reshetnyak in 1966—1968 [**29, 30, 31**]. In some sense it is a quasiconformal mapping admitting branch points. Later these mappings, under the name *quasiregular mappings*, were investigated intensively by Martio, Rickman, Väisälä, Gehring, Vuorinen, Bojarski, Iwaniec and others [**4, 12, 23, 24, 33, 37**].

The method of extremal lengths or the module of a family of curves was actively employed to treat analytic functions and quasiconformal mappings (see, for example, [**1, 2, 5, 38**]). Poletskiĭ successfully applied this method to study quasiregular mappings and obtained some interesting and fundamental results [**27, 28**].

Recently, the analysis on homogeneous groups has been developed intensively. Quasiconformal mappings on a homogeneous group of special type were initially considered by Mostow [**25**] in 1971 in connection with the rigidity theorems for the rank one symmetric space. Quasiconformal and quasiregular mappings on the Carnot groups have been studied, for instance, in [**8, 9, 14, 18, 35**].

The main result of this paper concerns with a characteristic of a singular set of quasiregular mappings. This singular set is defined in terms of the module of a family of locally rectifiable curves on Carnot groups. We prove that the module of a family of curves terminating on a closed set vanishes, if the module of a sub-family of this family, starting on a closed set of positive capacity, also vanishes. Precisely, let $\mathbb{G}$ be a Carnot group, $\Omega \subset \mathbb{G}$ be a domain, and $f : \Omega \to \mathbb{G}$ be a quasiregular mapping. Set I, A closed sets in Ω. By $\Gamma^\star(I)$ we denote the family of horizontal curves in $f(\Omega)$ admiting a lifting $\Gamma(I; \Omega)$ terminating on the set $I \subset \Omega$. We use the notation $\Gamma^\star(A; I)$ for the family of

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horizontal curves in $f(\Omega)$, such that the lifting of these curves $\Gamma(A, I; \Omega)$ starts on the set $A \subset \Omega$ and terminates on $I \subset \Omega$. We prove the next theorem.

THEOREM 1.1. *Let I, A be closed disjoint sets in $\Omega \subset \mathbb{G}$, such that* $\operatorname{cap} A > 0$. *Then $M(\Gamma^\star(I)) = 0$, if and only if $M(\Gamma^\star(A, I)) = 0$.*

In the next section the reader can find the exact definitions and preliminary results.

2. Definitions and preliminaries

The Carnot group is a connected and simply connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathcal{G}$ decomposes into the direct sum of vector subspaces $V_1 \oplus V_2 \oplus \ldots \oplus V_m$ satisfying the following relations:

$$[V_1, V_i] = V_{i+1}, \qquad 1 \leqslant i < m, \qquad [V_1, V_m] = \{0\}.$$

We identify the Lie algebra $\mathcal{G}$ with a space of left-invariant vector fields. Let $X_{11}, \ldots, X_{1n_1}$ be a basis of V_1, $n_1 = \dim V_1$, and $\langle \cdot, \cdot \rangle_0$ be a left-invariant Riemannian metric on V_1 such that

$$\langle X_{1i}, X_{1j} \rangle_0 = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Then, V_1 determines a subbundle HT of the tangent bundle $T\mathbb{G}$. We call HT the *horizontal tangent bundle* of $\mathbb{G}$ with HT_q as the *horizontal tangent space* at $q \in \mathbb{G}$. Respectively, the vector fields X_{1j}, $j = 1, \ldots, n_1$, are said to be *horizontal vector fields.*

Next, we extend $X_{11}, \ldots, X_{1n_1}$ to a basis X_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n_j = \dim V_i$, of $\mathcal{G}$. It is known (see, for instance, [**10**]) that the exponential map $\exp : \mathcal{G} \to \mathbb{G}$ from the Lie algebra $\mathcal{G}$ into the Lie group $\mathbb{G}$ is a global diffeomorphism. We can identify the points $q \in \mathbb{G}$ with the points $x \in \mathbb{R}^N$, $N = \sum\limits_{i=1}^{m} \dim V_i$, by means of the mapping $q = \exp\big(\sum\limits_{i,j} x_{ij} X_{ij}\big)$. The number $N = \sum\limits_{i=1}^{m} \dim V_i$ is the topological dimension of the Carnot group. The bi-invariant Haar measure on $\mathbb{G}$ is denoted by dx; this is the push-forward of the Lebesgue measure in $\mathbb{R}^N$ under the exponential map. *The family of dilations* $\{\delta_\lambda(x) : \lambda > 0\}$ on the Carnot group is defined as $\delta_\lambda x = \delta_\lambda(x_{ij}) = (\lambda^i x_{ij})$. Moreover, $d(\delta_\lambda x) = \lambda^Q dx$ and the quantity $Q = \sum\limits_{i=1}^{m} i \dim V_i$ is called *the homogeneous dimension* of $\mathbb{G}$.

EXAMPLE 1. The Euclidean space $\mathbb{R}^n$ with the standard structure is an example of an Abelian group. The exponential map is the identical mapping and the vector fields $X_i = \frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$, have only trivial commutators and constitute a basis for the corresponding Lie algebra.

EXAMPLE 2. The simplest example of a non-abelian Carnot group is the Heisenberg group $\mathbb{H}^n$. The non-commutative multiplication is defined as

$$pq = (x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2xy' + 2yx'),$$

where $x, x', y, y' \in \mathbb{R}^n$, $t, t' \in \mathbb{R}$. Left translation $L_p(\cdot)$ is defined as $L_p(q) = pq$. The left-invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \dots, n, \qquad T = \frac{\partial}{\partial t},$$

constitute the basis of the Lie algebra of the Heisenberg group. All non-trivial relations are only of the form $[X_i, Y_i] = -4T$, $i = 1, \dots, n$, and all other commutators vanish. Thus, the Heisenberg algebra has the dimension $2n + 1$ and splits into the direct sum $\mathcal{G} = V_1 \oplus V_2$. The vector space V_1 is generated by the vector fields X_i, Y_i, $i = 1, \dots, n$, and the space V_2 is the one-dimensional center which is spanned by the vector field T. For more information see [**17**].

EXAMPLE 3. A Carnot group is said to be of $\mathbb{H}$-type if the Lie algebra $\mathcal{G} = V_1 \oplus V_2$ is two-step and if the inner product $\langle \cdot, \cdot \rangle_0$ in V_1 can be extended to an inner product $\langle \cdot, \cdot \rangle$ in all of $\mathcal{G}$ so that the linear map $J : V_2 \to \mathrm{End}(V_1)$ defined by $\langle J_Z U, V \rangle = \langle Z, [U, V] \rangle$ satisfies $J_Z^2 = -\langle Z, Z \rangle \,\mathrm{Id}$ for all $Z \in V_2$. For the moment we introduce the notation $\|Z\|^2 = \langle Z, Z \rangle$. Then $\|J_Z V\| = \|Z\| \cdot \|V\|$ and $\langle V, J_Z V \rangle = 0$ for all $V \in V_1$ and $Z \in V_2$. More details and information see in [**7, 16**].

A homogeneous norm on $\mathbb{G}$ is, by definition, a continuous function $|\cdot|$ on $\mathbb{G}$ which is smooth on $\mathbb{G} \setminus \{0\}$ and such that $|x| = |x^{-1}|$, $|\delta_\lambda(x)| = \lambda |x|$, and $|x| = 0$ if and only if $x = 0$. The norm $|\cdot|$ defines a pseudo-distance: $d(x, y) = |x^{-1} y|$ satisfying the generalized triangle inequality $d(x, y) \leqslant \varpi(d(x, z) + d(z, y))$ with a positive constant ϖ. By $B(x, r)$ we denote an open ball in the metric d of radius $r > 0$ centered at x. By $\mathrm{mes}(E)$ we denote the measure of the set E. Our normalizing condition is such that the balls of radius one have measure one: $\mathrm{mes}(B(0,1)) = \int_{B(0,1)} dx = 1$. We have $\mathrm{mes}(B(0, r)) = r^Q$ because the Jacobian of the dilation δ_r is r^Q.

A continuous map $\gamma : I \to \mathbb{G}$ is called a curve. Here I is a (possibly unbounded) interval in $\mathbb{R}$. If $I = [a, b]$ then we say that $\gamma : [a, b] \to \mathbb{G}$ is a closed curve. A closed curve $\gamma : [a, b] \to \mathbb{G}$ is rectifiable if $\sup \Big\{ \sum_{k=1}^{p-1} d\big(\gamma(t_k), \gamma(t_{k+1})\big) \Big\} < \infty$, where the supremum ranges over all partitions $a = t_1 < t_2 < \ldots < t_p = b$ of the segment $[a, b]$. Pansu proved in [**26**] that any rectifiable curve is differentiable almost everywhere in (a, b) in the Riemannian sense and there exist measurable functions $a_j(s)$, $s \in (a, b)$, such that

$$\dot{\gamma}(s) = \sum_{j=1}^{n_1} a_j(s) X_{1j}(\gamma(s)) \quad \text{and} \quad d\big(\gamma(s + \tau), \gamma(s) \exp(\dot{\gamma}(s)\tau)\big) = o(\tau) \text{ as } \tau \to 0$$

for almost all $s \in (a, b)$. The length $l(\gamma)$ of a rectifiable curve $\gamma : [a, b] \to \mathbb{G}$ can be calculated by the formula

$$l(\gamma) = \int_a^b \langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle_0^{1/2} ds = \int_a^b \bigg(\sum_{j=1}^{n_1} |a_j(s)|^2 \bigg)^{1/2} ds$$

where $\langle \cdot, \cdot \rangle_0$ is the left invariant Riemannian metric on V_1. A result of [**6**] implies that one can connect two arbitrary points x, $y \in \mathbb{G}$ by a rectifiable curve. The Carnot-Carathéodory distance $d_c(x, y)$ is the infimum of the lengths over all rectifiable curves with endpoints x and $y \in \mathbb{G}$. The Hausdorff dimension of the metric space $(\mathbb{G}, d_c)$ coincides with the homogeneous dimension Q of the group $\mathbb{G}$.

DEFINITION 2.1. A function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{G}$, is said to be *absolutely continuous on lines* ($u \in \mathrm{ACL}(\Omega)$) if for any domain $U \Subset \Omega$, and any fibration $\mathcal{X}_j$ defined by the left-invariant vector fields X_{1j}, $j = 1, \dots, n_1$, the function u is absolutely continuous on $\gamma \cap U$ with respect to the $\mathcal{H}^1$-Hausdorff measure for $d\gamma$-almost all curves $\gamma \in \mathcal{X}_j$. (Recall that the measure $d\gamma$ on $\mathcal{X}_j$ equals the inner product $i(X_j)$ of the vector field X_j by the bi-invariant volume form dx.)

The Sobolev space $W_p^1(\Omega)$ ($L_p^1(\Omega)$), $1 \leqslant p < \infty$, consists of locally summable functions $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{G}$, having distributional derivatives $X_{1j}u$ along the vector fields X_{1j} and the finite norm

$$\|u \mid W_p^1(\Omega)\| = \Big(\int\limits_{\Omega} |u|^p\,dx\Big)^{1/p} + \Big(\int\limits_{\Omega} |\nabla_0 u|_0^p\,dx\Big)^{1/p}$$

$$\Big(\text{semi-norm} \qquad \|u \mid L_p^1(\Omega)\| = \Big(\int\limits_{\Omega} |\nabla_0 u|_0^p\,dx\Big)^{1/p}\Big).$$

Here $\nabla_0 u = (X_{11}u, \dots, X_{1n_1}u)$ is the *subgradient* of u and $|\nabla_0 u|_0 = \langle \nabla_0 u, \nabla_0 u\rangle_0$. We say, that u belongs to $W_{p,\mathrm{loc}}^1(\Omega)$ if for an arbitrary bounded domain U, $\overline{U} \subset \Omega$, the function u belongs to $W_p^1(U)$. For a function $u \in \mathrm{ACL}(\Omega)$, the derivatives $X_{1j}u$ along the vector fields X_{1j}, $j = 1, \dots, n_1$, exist almost everywhere in Ω. It is known that a function $u : \Omega \to \mathbb{R}$ belongs to $W_p^1(\Omega)$ ($L_p^1(\Omega)$), $1 \leqslant p < \infty$, if and only if it can be modified on a set of measure zero by such a way that $u \in L_p(\Omega)$ (u is locally p-summable), $u \in \mathrm{ACL}(\Omega)$, and $X_{1j}u \in L_p(\Omega)$ hold, $j = 1, \dots, n_1$.

DEFINITION 2.2. A mapping $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, belongs to the Sobolev class $W_{p,\mathrm{loc}}^1(\Omega)$, $1 \leqslant p < \infty$, if and only if it can be modified on a set of measure zero by such a way that

1) $|f(x)| \in L_{p,\mathrm{loc}}(\Omega)$;
2) the coordinate functions f_{ij} belong to $\mathrm{ACL}(\Omega)$ for all i and j;
3) $f_{1j} \in W_{p,\mathrm{loc}}^1(\Omega)$ for $1 \leqslant j \leqslant n_1$;
4) the vector $X_{1k}(f(x)) = \sum\limits_{1 \leqslant l \leqslant m, 1 \leqslant \omega \leqslant n_l} X_{1k}(f_{l\omega}(x))\frac{\partial}{\partial x_{l\omega}}$ belongs to $HT_{f(x)}$ for almost all $x \in \Omega$ and all $k = 1, \dots, n_1$.

In [**13, 36**], one can find various definitions of the Sobolev space on Carnot groups and their correlations. The matrix $X_{1k}f = (X_{1k}f_{1j})_{k,j=1,\dots,n_1}$ defines a linear operator $D_H f : V_1 \to V_1$ [**26**] which is called a *formal horizontal differential*. A norm of the operator $D_H f$ is defined by

$$|D_H f(x)| = \sup_{\xi \in V_1, |\xi|_0 = 1} |D_H f(x)(\xi)|_0.$$

The norm $|D_H f|$ is equivalent to $|\nabla_0 f|_0 = \Big(\sum\limits_{i=1}^{n_1} |X_{1i}f|_0^2\Big)^{\frac{1}{2}}$. It has been proved in [**36**] that the formal horizontal differential $D_H f$ generates a homomorphism $Df : \mathcal{G} \to \mathcal{G}$ of

Lie algebras which is called a *formal differential*. The determinant of the matrix $Df(x)$ is denoted by $J(x, f)$ and called a (*formal*) *Jacobian*.

A continuous map $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, is *open* if the image of an open set is open and *discrete* if the pre-image $f^{-1}(y)$ of each point $y \in f(\Omega)$ consists of isolated points. We say that f is sense-preserving if a topological degree $\mu(y, f, U)$ is strictly positive for all domains U, $\overline{U} \subset \Omega$ and $y \in f(U) \setminus f(\partial U)$.

Definition 2.3. Let Ω be a domain on the group $\mathbb{G}$. A mapping $f : \Omega \to \mathbb{G}$ is said to be a *quasiregular mapping* if

1) f is continuous open discrete and sense-preserving ;
2) f belongs to $W^1_{Q,\mathrm{loc}}(\Omega)$;
3) the formal horizontal differential $D_H f$ satisfies the condition

$$\max_{|\xi|_0=1, \xi\in V_1} |D_H f(x)(\xi)|_0 \leqslant K \min_{|\xi|_0=1, \xi\in V_1} |D_H f(x)(\xi)|_0 \tag{2.1}$$

for almost all $x \in \Omega$.

It is known [**36**] that the pointwise inequality (2.1) is equivalent to the following one: *the formal horizontal differential $D_H f$ satisfies the condition*

$$|D_H f(x)|^Q \leqslant K' J(x, f) \tag{2.2}$$

for almost all $x \in \Omega$ where K' depends on K. The smallest constant K' in inequality (2.2) is called the *outer distortion* and denoted by $K_O(f)$. It is not hard to see that for a quasiregular mapping the inequality

$$0 \leqslant J(x, f) \leqslant K'' \min_{|\xi|_0=1, \xi\in V_1} |D_H f(x)(\xi)|_0^Q \tag{2.3}$$

also holds for almost all $x \in \Omega$ where K'' depends on K. The smallest constant K'' in inequality (2.3) is called the *inner distortion* and denoted by $K_I(f)$.

Definition 2.4. A continuous mapping $f : \Omega \to \mathbb{G}$ is $\mathcal{P}$-differentiable at $x \in \Omega$ if the family of maps $f_t = \delta_{1/t}(f(x)^{-1} f(x\delta_t y))$ converges locally uniformly to an automorphism of $\mathbb{G}$ as $t \to 0$.

In the following theorem we formulate some analytic properties of quasiregular mappings [**35, 36**]. We denote by B_f the set of points where a quasiregular mapping f is not homeomorphic. In the statement of the theorem we use notions of the topological degree $\mu(y, f, D)$ of the mapping f and the multiplicity function $N(y, f, A) = \mathrm{card}\{x \in f^{-1}(y) \cap A\}$ (see the precise definitions, for instance, in [**34**]).

Theorem 2.1. *Let $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, be a quasiregular mapping. Then it possesses the following properties:*

1) *f is $\mathcal{P}$-differentiable almost everywhere in Ω;*
2) *$\mathcal{N}$-property: if* $\mathrm{mes}(A) = 0$ *then* $\mathrm{mes}(f(A)) = 0$;
3) *$\mathcal{N}^{-1}$-property: if* $\mathrm{mes}(A) = 0$ *then* $\mathrm{mes}(f^{-1}(A)) = 0$;
4) $\mathrm{mes}(B_f) = \mathrm{mes}(f(B_f)) = 0$;
5) *$J(x, f) > 0$ almost everywhere in Ω;*

6) *for every compact domain* $D \Subset \Omega$ *such that* $\operatorname{mes}(f(\partial D)) = 0$ *(every measurable set* $A \subset \Omega$*) and every measurable function* u*, the function* $y \mapsto u(y)\mu(y, f, D)$ $(y \mapsto u(y)N(y, f, D))$ *is integrable in* $\mathbb{G}$ *if and only if the function* $(u \circ f)(x)J(x, f)$ *is integrable on* D (A)*; moreover, the following change of variable formulas hold:*

$$\int\limits_D (u \circ f)(x)J(x, f)\, dx = \int\limits_{\mathbb{G}} u(y)\mu(y, f, D)\, dy, \tag{2.4}$$

$$\int\limits_A (u \circ f)(x)J(x, f)\, dx = \int\limits_{\mathbb{G}} u(y)N(y, f, A)\, dy. \tag{2.5}$$

If A is a closed set in an open set $\Omega \in \mathbb{G}$, then we use the following definition of the capacity:

$$\operatorname{cap} A = \inf \int\limits_{\mathbb{G}} |\nabla_0 v|^Q\, dx,$$

where the infimum is taken over all non-negative functions $v \in C_0^\infty(\Omega)$, such that $v|_A \geqslant 1$.

The linear integral is defined by $\int_\gamma \rho\, ds = \sup \int_{\gamma'} \rho\, ds = \sup \int_0^{l(\gamma')} \rho(\gamma'(s))\, ds$, where the supremum is taken over all closed parts γ' of γ and $l(\gamma')$ is the length of γ'. Let Γ be a family of curves in $\mathbb{G}$. Denote by $\mathcal{F}(\Gamma)$ the set of Borel functions $\rho : \mathbb{G} \to [0; \infty]$, such that the inequality $\int\limits_\gamma \rho\, ds \geqslant 1$ holds for locally rectifiable curves $\gamma \in \Gamma$.

Definition 2.5. Let Γ be a family of curves in $\overline{\mathbb{G}}$. The quantity

$$M(\Gamma) = \inf \int\limits_{\mathbb{G}} \rho^Q\, dx$$

is called the *module of the family of curves* Γ. The infimum is taken over all functions $\rho \in \mathcal{F}(\Gamma)$.

Here and subsequently $\langle a, b\rangle$ stands for an interval of one of the following type (a, b), $[a, b)$, $(a, b]$, and $[a, b]$. Let F_0, F_1 be disjoint compacts in $\overline{\Omega}$. We say that a curve $\gamma : \langle a, b\rangle \to \Omega$ connects F_0 and F_1 in Ω (terminates on F_0 in Ω) if

1. $\overline{\gamma(\langle a, b\rangle)} \cap F_i \neq \emptyset$, $i = 0, 1$, $(\overline{\gamma(\langle a, b\rangle)} \cap F_0 \neq \emptyset)$,
2. $\gamma(t) \in \Omega$ for all $t \in (a, b)$.

A family of curves connecting F_0 and F_1 (terminating at F_0) in Ω is denoted by $\Gamma(F_0, F_1; \Omega)$ $(\Gamma(F_0; \Omega))$.

Remark 2.1. Let $f : \Omega \to \mathbb{G}$ be a quasiregular mapping and Γ be a family of curves in Ω. We correlate the parametrization of the curves in $\Gamma \subset \Omega$ and in $\Gamma^\star = f(\Gamma) \subset f(\Omega)$. Let $\gamma^\star \in \Gamma^\star$ be a rectifiable curve. We introduce the length arc parameter $s^\star$ in the curve $\gamma^\star \in \Gamma^\star$. Thus $s^\star \in I^\star = [0, l(\gamma^\star)]$ where $l(\gamma^\star)$ is the length of the curve $\gamma^\star$. If t is any other parameter on $\gamma^\star$: $\gamma^\star(t) = f(\gamma(t))$, then the function $s^\star(t)$ is strictly monotone and continuous, so the same holds for its inverse function $t(s^\star)$. For the curve $\gamma(t) \in \Gamma$, such that $f(\gamma(t)) = \gamma^\star$, the parameter $s^\star$ can be introduced by the following way

$$f(\gamma(t(s^\star))) = f(\gamma(s^\star)) = \gamma^\star(s^\star), \quad s^\star \in I^\star.$$

We note that if we take the length arc parameter s on γ, $s \in I = [0, l(\gamma)]$ and the length arc parameter $s^\star$ on $\gamma^\star$, $s^\star \in I^\star = [0, l(\gamma^\star)]$, then

$$1 = \Big|\frac{d\gamma(s)}{ds}\Big|_0 = \Big|\frac{d\gamma(s^\star)}{ds^\star}\Big|_0 \cdot \Big|\frac{ds^\star}{ds}\Big|. \tag{2.6}$$

and

$$1 = \Big|\frac{d\gamma^\star(s^\star)}{ds^\star}\Big|_0 = \Big|\frac{d\gamma^\star(s)}{ds}\Big|_0 \cdot \Big|\frac{ds}{ds^\star}\Big|. \tag{2.7}$$

From now on, we use the letters s and $s^\star$ to denote the length arc parameters on curves $\gamma \in \Gamma$ and $\gamma^\star \in \Gamma^\star$. The corresponding domains of s and $s^\star$ are denoted by $I = [0, l(\gamma)]$ and $I^\star = [0, l(\gamma^\star)]$, respectively.

We state here a Poletskiĭ type lemma. Its complete proof for $\mathbb{R}^n$ can be found in [**27, 33**] and for Carnot groups in [**19, 20**].

Lemma 2.1. *Let $f : \Omega \to \mathbb{G}$ be a non-constant quasiregular mapping and $U \subset \Omega$ be a domain, such that $\overline{U} \subset \Omega$. Assume Γ to be a family of curves in U such that $\gamma^\star(s^\star) = f(\gamma(s^\star))$ is locally rectifiable and there exists a closed part $\gamma'(s^\star)$ of $\gamma(s^\star)$ that is not absolutely continuous (the parameterization of Γ and $f(\Gamma)$ is correlated as in Remark 2.1). Then, $M(f(\Gamma)) = 0$*

Let $f : \Omega \to \mathbb{G}$ be a continuous discrete and open mapping of a domain $\Omega \in \mathbb{G}$. Let $\beta : [a, b[\in \mathbb{G}$ be a curve and let $x \in f^{-1}(\beta(a))$. A curve $\alpha : [a, c[\to \Omega$ is called an *f-lifting of β starting at point x* if

1) $\alpha(a) = x$,
2) $f \circ \alpha = \beta|_{[a,c[}$,.

We say that a curve $\alpha : [a, c[\to \Omega$ is a *maximal* f-lifting of β starting at point x if both 1), 2) and the following property hold:

3) if $c < c' < b$ then there does not exist a curve $\alpha' : [a, c'[\to \Omega$ such that $\alpha = \alpha'|_{[a,c[}$ and $f \circ \alpha' = \beta|_{[a,c'[}$.

Let $f^{-1}(\beta(a)) = \{x_1, \dots, x_k\}$ and $m = \sum\limits_{j=1}^{k} i(x_j, f)$. We say that $\alpha_1, \dots, \alpha_m$ is a *maximal essentially separate* sequence of f-liftings of β starting at the points $x_1, \dots, x_k$ if

1) each α_j is a maximal lifting of f,
2) $\mathrm{card}\{j : \alpha_j(a) = x_l\} = i(x_l, f)$, $1 \leqslant l \leqslant k$,
3) $\mathrm{card}\{j : \alpha_j(t) = x\} \leqslant i(x, f)$ for all $x \in \Omega$ and all t.

Similarly, we define a maximal sequence of f-liftings terminating at $x_1, \dots, x_k$ if $f :]b, a] \to \mathbb{G}$. More information about the existence and the properties of liftings can be found in [**32, 40**].

The next statement is a generalization of the inequality of Väisälä. The Väisälä inequality is an essential tool on the study of quasiregular mappings. For proof of this inequality see [**22, 33**].

THEOREM 2.2. *Let* $f : \Omega \to \mathbb{G}$ *be a nonconstant quasiregular mapping,* Γ *be a family of curves in* Ω*,* $\Gamma^\star$ *be a family in* $\mathbb{G}$ *and* m *be a positive integer such that the following is true. For every locally rectifiable curve* $\beta : \langle a, b\rangle \to \mathbb{G}$ *in* $\Gamma^\star$ *there exist curves* $\alpha_1, \dots \alpha_m$ *in* Γ *such that*

1) $(f \circ \alpha_j) \subset \beta$ *for all* $j = 1, \dots, m$*,*
2) $\operatorname{card}\{j : \ \alpha_j(t) = x\} \leqslant i(x, f)$ *for all* $x \in \Omega$ *and for all* $t \in \langle a, b\rangle$*.*

Then

$$M(\Gamma^\star) \leqslant \frac{K_I(f)}{m} M(\Gamma).$$

3. Proof of the principal results

In the statement of the theorem we use the following notations. A Carnot group is denoted by $\mathbb{G}$, Ω is a domain on $\mathbb{G}$, $f : \Omega \to \mathbb{G}$, is a quasiregular mapping. Let I and A be closed sets in a domain Ω. By $\Gamma^\star(I)$ we denote the family of locally rectifiable curves in $f(\Omega)$ that admit maximal essentially separate liftings $\Gamma(I; \Omega)$ terminating on the set $I \subset \Omega$. Let $\Gamma^\star(A, I)$ be a family of locally rectifiable curves in $f(\Omega)$ such, that the maximal essentially separate liftings of these curves $\Gamma(A, I; \Omega)$ start on the set $A \subset \Omega$ and terminate in $I \subset \Omega$. We recall the statement of the principal theorem.

THEOREM 1.1 *Let* I*,* A *be closed disjoint sets in* $\Omega \subset \mathbb{G}$*, such that* $\operatorname{cap} A > 0$*. Then* $M(\Gamma^\star(I)) = 0$*, if and only if* $M(\Gamma^\star(A, I)) = 0$*.*

Proof of Theorem 1.1. Since $\Gamma^\star(A, I) \subset \Gamma^\star(I)$, we have $M(\Gamma^\star(A, I)) \leqslant M(\Gamma^\star(I))$ and the necessary part is obvious.

Let us prove that the assumption $M(\Gamma^\star(A, I)) = 0$ implies $M(\Gamma^\star(I)) = 0$. We consider an r-neighborhood I_r of the set I and a set G, such that $G = A \cap (\Omega \setminus \overline{I}_{2r})$ and $\operatorname{cap} G > 0$. We fix $\varepsilon \in (0, 1)$ and choose an admissible function $\rho^\star(y)$ for the family $\Gamma^\star(A, I)$, such that $\int_{f(\Omega)} (\rho^\star(y))^Q \, dy < \varepsilon$. We denote by E, $E \subset \Omega$, the set of points where the mapping f is not $\mathcal{P}$-differentiable. There exists a Borel set F of measure zero, such that $E \cup B_f \subset F$. Let us define a function $\rho(x)$ on Ω by the rule

$$\rho(x) = \begin{cases} \rho^\star(f(x)) \cdot |D_H f(x)| & \text{if} \quad x \in \Omega \setminus (I \cup F), \\ 0 & \text{if} \quad x \in I \cup F. \end{cases} \tag{3.1}$$

We claim that the function $\rho(x)$ is admissible for the family of curves $\Gamma(A, I; \Omega)$. Indeed, if $\gamma \in \Gamma(A, I; \Omega)$ is a lifting of a curve $\gamma^\star \in \Gamma^\star(A, I)$ and $s \in I$, $s^\star \in I^\star$ are the arc length parameters of curves γ and $\gamma^\star$ respectively, then we obtain

$$\begin{aligned} \int_\gamma \rho \, ds &= \int_I \rho^\star(f(\gamma(s)))|D_H f(\gamma(s))| \, ds = \int_{I^\star} \rho^\star(f(\gamma(s^\star)))|D_H f(\gamma(s^\star))| \Big|\frac{ds}{ds^\star}\Big| \, ds^\star \\ &= \int_{I^\star} \rho^\star(\gamma^\star(s^\star))|D_H(\gamma^\star(s^\star)| \Big|\frac{d\gamma^\star}{ds}\Big|_0^{-1} ds^\star \geqslant \int_{I^\star} \rho^\star(\gamma^\star(s^\star)) \, ds^\star \\ &= \int_{\gamma^\star} \rho^\star \, ds^\star \geqslant 1 \end{aligned}$$

by (2.7) and the inequality $|D_H(\gamma^\star(s^\star)| \big|\frac{d\gamma^\star}{ds}\big|_0^{-1} \geqslant 1$.

Two subsets $C_\varepsilon^{(r)}$ and $D_\varepsilon^{(r)}$of the boundary ∂I_r are considered. Denote by $C_\varepsilon^{(r)}$ the set of the points $x \in \partial I_r$ for which there exists a curve $\alpha \in \Gamma(A, I; \Omega)$ passing through x and satisfying the condition: $\int\limits_{\tilde{\alpha}} \rho\, ds < 1/2$ for an arc $\tilde{\alpha}$ of the curve α such that $\tilde{\alpha} \in \Omega \setminus \overline{I}_r$. Since the function ρ is admissible, we deduce that for any curve α that starts at $x \in C_\varepsilon^{(r)}$ and terminates on I we have $\int\limits_{\alpha} \rho\, ds \geqslant 1/2$. Thus, 2ρ is an admissible function for $\Gamma(C_\varepsilon^{(r)}, I; \Omega)$.

The subset $D_\varepsilon^{(r)}$ is the complement to $C_\varepsilon^{(r)}$: $D_\varepsilon^{(r)} = \partial I_r \setminus C_\varepsilon^{(r)}$. By definition of $D_\varepsilon^{(r)}$, for any $\gamma \in \Gamma(G, D_\varepsilon^{(r)}; \Omega \setminus \overline{I}_r)$, we get $\int\limits_{\gamma} \rho\, ds \geqslant 1/2$. We deduce

$$
\begin{aligned}
M(\Gamma(G, D_\varepsilon^{(r)}; \Omega \setminus \overline{I}_r)) &\leqslant 2^Q \int\limits_{\Omega\setminus\overline{I}_r} \rho^Q\, dx \leqslant 2^Q \int\limits_{\Omega} \big(\rho^\star(f(x))\big)^Q |D_H f(x)|^Q\, dx \\
&\leqslant 2^Q K_O(f) \int\limits_{\Omega} \big(\rho^\star(f(x))\big)^Q J(x,f)\, dx \qquad (3.2)\\
&= 2^Q K_O(f) \int\limits_{f(\Omega)} \big(\rho^\star\big)^Q N(y, f, \Omega \setminus \overline{I}_r)\, dy \leqslant 2^Q K_O(f) N \varepsilon,
\end{aligned}
$$

where $N = \sup\limits_{y \in \mathbb{G}} N(y, f, \Omega \setminus \overline{I}_r)$.

Let us estimate the module of the family of curves $\Gamma^\star(C_\varepsilon^{(r)}, I) \subset \Gamma^\star(I)$ whose lifting starts at $C_\varepsilon^{(r)}$ and terminates at I. We denote $\lambda_f(x) = \min\limits_{|\xi|_0 = 1, \xi \in V_1} |D_H f(x)(\xi)|_0$. If x belongs to $\Omega \setminus (I \cup F)$, then for a function $\rho^\star \in \mathcal{F}\big(\Gamma^\star(C_\varepsilon^{(r)}, I)\big)$, we get

$$
\rho^\star(y) = \rho^\star(f(x)) = \frac{\rho(x)}{|D_H f(x)|} \geqslant \frac{\rho(x)}{K_O^{1/Q} J^{1/Q}(x,f)} \geqslant \frac{\rho(x)}{K_O^{1/Q} K_I^{1/Q} \lambda_f(x)} \qquad (3.3)
$$

from (2.2) and (2.3). It can be proved, that since $\operatorname{mes}(f(F)) = 0$, we have $\int\limits_{\gamma^\star} \chi_{f(F)}\, ds^\star = 0$ for $\gamma^\star \in \Gamma^\star(C_\varepsilon^{(r)}, I)$ and characteristic function $\chi_{f(F)}$ of the set $f(F)$ (see [**22, 39**]). Thus,

$$
\begin{aligned}
\int\limits_{\gamma^\star} \rho^\star(s^\star)\, ds^\star &= \int\limits_{I^\star} \rho^\star(\gamma^\star(s^\star))\, ds^\star = \int\limits_{I} \rho^\star(f(\gamma(s))) \Big|\frac{ds^\star}{ds}\Big|\, ds \\
&\geqslant K_O^{-\frac{1}{Q}}(f) K_I^{-\frac{1}{Q}}(f) \int\limits_{I} \rho(\gamma(s)) \Big(\lambda_f(\gamma(s)) \Big|\frac{d\gamma(s^\star)}{ds^\star}\Big|_0\Big)^{-1} ds \\
&\geqslant \frac{1}{K_O^{1/Q}(f) K_I^{1/Q}(f)} \int\limits_{\gamma} \rho(s)\, ds \geqslant \frac{1}{2K_O^{1/Q}(f) K_I^{1/Q}(f)}
\end{aligned}
$$

by (3.3), (2.6), and the inequality $\big(\lambda_f(\gamma(s)) \big|\frac{d\gamma(s^\star)}{ds^\star}\big|_0\big)^{-1} \geqslant 1$.

Finally, we deduce

$$M(\Gamma^\star(C_\varepsilon^{(r)}, I)) \leqslant 2^Q K_O(f) K_I(f) \int\limits_{f(\Omega)} \left(\rho^\star\right)^Q dy \leqslant 2^Q K_O(f) K_I(f)\varepsilon. \tag{3.4}$$

Now we choose the sequence $\varepsilon_l = (2^{Q+l} K_O(f) K_I(f) j)^-1$, $l, j \in \mathbb{N}$. For the union $C_j^{(r)} = \bigcup\limits_{l=1}^{\infty} C_{\varepsilon_l}^{(r)}$ we obtain

$$M(\Gamma^\star(C_j^{(r)}, I)) \leqslant \sum_{l=1}^{\infty} M(\Gamma^\star(C_{\varepsilon_l}^{(r)}, I)) \leqslant \frac{1}{j} \sum_{l=1}^{\infty} \frac{1}{2^l} \leqslant \frac{1}{j} \tag{3.5}$$

from (3.4) and from the subadditivity of the module of a family of curves. For the set $D_j^{(r)} = \bigcap\limits_{l=1}^{\infty} D_{\varepsilon_l}^{(r)}$ from (3.2), we have

$$M(\Gamma(G, D_j^{(r)}; \Omega \smallsetminus \overline{I}_r)) = 0. \tag{3.6}$$

The estimates (3.5) and (3.6) imply that

$$M(\Gamma(G, D^{(r)}; \Omega \smallsetminus \overline{I}_r)) = 0 \qquad \text{with} \qquad D^{(r)} = \bigcup_{j=1}^{\infty} D_j^{(r)},$$

$$M(\Gamma^\star(C^{(r)}, I)) = 0 \qquad \text{with} \qquad C^{(r)} = \bigcap_{j=1}^{\infty} C_j^{(r)},$$

and

$$C^{(r)} \cup D^{(r)} = \partial I_r.$$

The next step of our proof is to show that $M(\Gamma(D^{(r)}; \Omega \smallsetminus \overline{I}_r)) = 0$, where $\Gamma(D^{(r)}; \Omega \smallsetminus \overline{I}_r)$ is the family of curves connecting the points $x \in \Omega \smallsetminus \overline{I}_r$ with the set $D^{(r)}$. Since $M(\Gamma(G, D^{(r)}; \Omega \smallsetminus \overline{I}_r)) = 0$, we can choose a function $\rho \in L_Q(\Omega)$, such that $\int\limits_\gamma \rho\, ds = \infty$ for any curve $\gamma \in \Gamma(G, D^{(r)}; \Omega \smallsetminus \overline{I}_r)$. Making use of constructions from [**3, 15, 21**] we can suppose that ρ is continuous in $\Omega \smallsetminus \overline{I}_r$.

Now, let P be a subset of $\mathcal{Q} = \Omega \smallsetminus (\overline{I}_r \cup G)$ with the following property: there is a curve $\gamma \in \Gamma(P, G; \Omega \smallsetminus \overline{I}_r)$, such that $\int\limits_\gamma \rho(s)\, ds < \infty$. We claim that P is open and close in $\mathcal{Q}$. First, we show that P is open. Let $x \in P$ and $B(x, \frac{\delta}{2})$ be a ball in $\mathcal{Q}$ such that $B(x, \delta) \in \mathcal{Q}$. We choose a point $\omega \in B(x, \frac{\delta}{2})$ and we connect ω with x by a rectifiable curve α. The function ρ is locally bounded, therefore $\int\limits_\alpha \rho\, ds < \infty$. Thus,

$$\int\limits_{\gamma \cup \alpha} \rho\, ds = \int\limits_\gamma \rho\, ds + \int\limits_\alpha \rho\, ds < \infty, \qquad \gamma \in \Gamma(x, G; \Omega \smallsetminus \overline{I}_r),$$

and we deduce that P is open.

We note that $\int\limits_\gamma \rho(s)\, ds = \infty$ for any $\gamma \in \Gamma(P, D^{(r)}; \Omega \smallsetminus \overline{I}_r)$. If it were not so, then we could choose a curve $\widetilde{\gamma} \in \Gamma(P, G; \Omega \smallsetminus \overline{I}_r)$, such that $\int\limits_{\widetilde{\gamma}} \rho(s)\, ds < \infty$ and get a contradiction with $\int\limits_{\gamma\cup\widetilde{\gamma}} \rho(s)\,ds = \infty$, where the curve $\gamma \cup \widetilde{\gamma}$ connects G and $D^{(r)}$. Finally, we have

$$M(\Gamma(P, D^{(r)}; \Omega \setminus \overline{I}_r)) = 0. \tag{3.7}$$

We assume that $P \neq \emptyset$ and show that P is closed in $\mathcal{Q}$. Let x be a limit point of the set P. Let us take a sufficiently small ball $B(x,\delta)$, $\overline{B}(x,\delta) \subset \mathcal{Q}$, and connect x with some point $x' \in B(x, \frac{\delta}{2}) \cap P$ by a rectifiable curve β, that belongs to $\mathcal{Q} \cap \overline{B}(x,\delta)$. Since ρ is continuous in $\mathcal{Q}$, then it is bounded in $\overline{B}(x,\delta)$ and $\int\limits_{\beta} \rho(s)\,ds < \infty$. The point x' belongs to P, hence there is a curve $\gamma \in \Gamma(x', G; \Omega \setminus \overline{I}_r)$ such that $\int\limits_{\gamma} \rho(s)\,ds < \infty$. Consequently, we have $\int\limits_{\gamma\cup\beta} \rho(s)\,ds < \infty$ for the curve that connect x and G. The point x belongs to P, it means that P is closed.

By the next step we show that the complement $\mathcal{Q} \setminus P$ is empty. From the contrary, let us assume that $H = \mathcal{Q} \setminus P$ is not empty. We denote by $\mathcal{Q}_i$ connected components of $\mathcal{Q}$. Since $H = \mathcal{Q} \setminus P$ is open and closed, the components $\mathcal{Q}_i$ lie either in $H = \mathcal{Q} \setminus P$ or in P. If $\mathcal{Q}_i \subset P$, then $M(\Gamma(\mathcal{Q}_i, D^{(r)}; \Omega \setminus \overline{I}_r)) = 0$. If $\mathcal{Q}_i \subset \mathcal{Q} \setminus P$, then we can choose a ball $B_0 = B(x, \varrho) \subset \mathcal{Q}_i$ such that $\int\limits_{\gamma} \rho(s)\,ds = \infty$ for any $\gamma \in \Gamma(B_0, G; \Omega \setminus \overline{I}_r)$. Consequently, $M(\Gamma(B_0, G; \Omega \setminus \overline{I}_r) = 0$.

We denote by W the set of points from $\Omega \setminus \overline{I}_r$ such that there is no rectifiable curve joining W with B_0 which does not intersect G. It is obvious, that W contains G. This and a result by B. Fuglede [**11**] imply that $M(\Gamma(B_0, W; \Omega \setminus \overline{I}_r)) = 0$.

The set W is closed. Really, if we choose $x' \in \complement W$, then there exists a rectifiable curve γ connecting x' and B_0. Let $B(x', \epsilon)$ be a small ball, $x'' \in B(x', \frac{\epsilon}{2})$. We unite x' and x'' by a rectifiable curve α. Since the function $\rho(x)$ is continuous in $\Omega \setminus \overline{I}_r$ we obtain $\int\limits_{\alpha} \rho(s)\,ds < \infty$ and $\int\limits_{\alpha\cup\gamma} \rho(s)\,ds < \infty$. So the set W is closed.

Let us show that $M(\Gamma(W, \mathcal{Q}_i \setminus W; \Omega \setminus \overline{I}_r)) = 0$. If $y \in \complement W$ and $\gamma \in \Gamma(y, W; \Omega \setminus \overline{I}_r)$, then $\int\limits_{\gamma} \rho(s)\,ds = \infty$. Suppose that it is not so: $\int\limits_{\gamma} \rho(s)\,ds < \infty$. We connect y and B_0 by a rectifiable curve γ'. The continuity of the function ρ implies $\int\limits_{\gamma'\cup\gamma} \rho(s)\,ds < \infty$. This contradicts to the fact that $M(\Gamma(B_0, W; \Omega \setminus \overline{I}_r)) \leqslant M(\Gamma(B_0, G; \Omega \setminus \overline{I}_r)) = 0$. Hence, $M(\Gamma(W, \mathcal{Q}_i \setminus W; \Omega \setminus \overline{I}_r)) = 0$. This implies $\operatorname{cap} W = 0$, that contradicts to $\operatorname{cap} G = 0$.

We have shown that $H = \emptyset$ and, consequently, $P = \mathcal{Q}$. Finally,

$$M(\Gamma(D^{(r)}; \Omega \setminus \overline{I}_r)) = 0,$$

where $\Gamma(D^{(r)}; \Omega \setminus \overline{I}_r)$ is a family of curves joining points $x \in \Omega \setminus \overline{I}_r$ with $D^{(r)}$. We choose a sequence $r_k \to 0$ as $k \to \infty$. Any curve $\gamma^\star \in \Gamma^\star(I)$ has a maximal essentially separate lifting $\alpha_1, \ldots, \alpha_j$ that starts on $\Omega \setminus \overline{I}_{r_k}$ for some k. Since $\Omega \setminus I$ is connected, we can choose k sufficiently big, such that starting point of the lifting lies in a connected component of $\Omega \setminus \overline{I}_{r_k}$ with $\operatorname{cap}(A \cap (\Omega \setminus \overline{I}_{r_k})) > 0$. This lifting intersects either the set C^{r_k} or D^{r_k}. In the first case we have $M(\Gamma^\star(C^{(r_k)}, I)) = 0$. In the second one

$M(\Gamma(D^{(r_k)};\Omega\setminus\overline{I}_{r_k}))=0$ and Theorem 2.2 implies that

$$M(\Gamma^\star(D^{(r_k)}))\leqslant\frac{K_I(f)}{m}M(\Gamma(D^{(r_k)};\Omega\setminus\overline{I}_{r_k}))=0.$$

So $M(\Gamma^\star(C^{(r_k)},I)\cup\Gamma^\star(D^{(r_k)}))=0$. Finally, letting $k\to\infty$ we deduce

$$M(\Gamma^\star(I))=0.$$

□

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