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On the Lambek Invariants of Commutative Squares in a Quasi-Abelian Category

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Abstract. We consider the invariants Ker and Im for commutative squares in quasi-abelian categories. These invariants were introduced by Lambek for groups and then studied by Hilton and Nomura in categories exact in the sense of Buchsbaum.

1. Introduction

In 1964, Lambek [**13**] introduced the following invariants for a commutative square

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & D \\ {\scriptstyle g}\downarrow & S & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\beta} & B \end{array} \tag{1.1}$$

in the category of groups:

$$\operatorname{Im} S = (\operatorname{Im}\beta \cap \operatorname{Im} f)/\operatorname{Im}(f\alpha), \quad \operatorname{Ker} S = \operatorname{Ker}(f\alpha)/(\operatorname{Ker}\alpha + \operatorname{Ker} g).$$

In [**13**], he proved the following assertion.

Given a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle a}\downarrow & S & {\scriptstyle b}\downarrow & T & \downarrow{\scriptstyle c} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array} \tag{1.2}$$

of groups and group homomorphisms with exact rows, there is a natural isomorphism

$$\Lambda : \operatorname{Im} S \xrightarrow{\cong} \operatorname{Ker} T.$$

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Later Leicht [**14**] extended this theorem to arbitrary Buchsbaum-exact categories [**1, 15**]. In [**16, 17**], Nomura considered the case where the rows in (1.2) are not exact but only semiexact, constructed a canonical morphism $\Lambda : \operatorname{Im} S \longrightarrow \operatorname{Ker} T$, and proved that there is an exact sequence

$$(1.3)\quad 0 \to H(\operatorname{Ker}(bf) \to \operatorname{Ker} b \to \operatorname{Ker} c) \to \operatorname{Ker}(H \to H') \to \operatorname{Im} S \xrightarrow{\Lambda} \operatorname{Ker} T \\ \to \operatorname{Coker}(H \to H') \to H(\operatorname{Coker} a \to \operatorname{Coker} b \to \operatorname{Coker}(g'b)) \to 0,$$

where the arrows between the kernels and cokernels in parentheses are natural morphisms, $H(\cdot \to \cdot \to \cdot)$ stands for the homology of the 0-sequence in parentheses, $H = H(A \to B \to C)$, and $H' = H(A' \to B' \to C')$. Later Ubeda Bescansa generalized Nomura's results to what he called *categorías Hofmanianas* [**28, 29**], a special case of a *homological monoid* [**7**].

In this paper, we study the Lambek invariants in quasi-abelian categories, first considered by Raĭkov in [**22**] under the name of semiabelian categories. Apart from all abelian categories, the class of quasi-abelian categories contains many nonabelian additive categories of functional analysis and topological algebra. The categories of (Hausdorff or all) topological abelian groups, topological vector spaces, Banach (or normed) spaces, filtered modules over filtered rings, and torsion-free abelian groups are typical examples of quasi-abelian categories. The main difference between the quasi-abelian and abelian categories lies in the fact that the standard diagram lemmas hold in quasi-abelian categories under some extra conditions which usually amount to the strictness of some morphisms. Quasi-abelian categories have been actively studied in the recent years (see [**4, 5, 9, 10, 11, 19, 20, 21, 23, 24, 25, 26**]).

In the category $\mathcal{B}an$ of Banach spaces topological abelian groups, the strictness of a morphism α means that the range of α is closed. In the category of topological abelian groups, a morphism α is strict if and only if its image is closed and, moreover, α maps open sets onto open sets.

In a quasi-abelian category, Nomura's morphism $\Lambda : \operatorname{Im} S \longrightarrow \operatorname{Ker} T$ is defined only if b is strict in (1.2) because the definition uses the fact that b is the composition of its image and coimage. Lambek's isomorphism holds under the same condition (see [**16**]).

The structure of the paper is as follows. In Section 2, we recall some basic definitions and facts about quasi-abelian categories. In Section 3, we construct a morphism $\zeta : \operatorname{Ker} T \longrightarrow \operatorname{Im} S$ for a diagram (1.2) with exact rows in the general case and suggest quasi-abelian versions for some assertions proved by Nomura [**16**] and Hilton [**6**] for abelian and Buchsbaum-exact categories.

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2. Quasi-Abelian Categories

We consider additive categories satisfying the following axiom.

AXIOM 1. Each morphism has kernel and cokernel.

We denote by $\ker\alpha$ ($\operatorname{coker}\alpha$) an arbitrary kernel (cokernel) of α and by $\operatorname{Ker}\alpha$ ($\operatorname{Coker}\alpha$) the corresponding object; the equality $a = \ker b$ ($a = \operatorname{coker} b$) means that a is a kernel of b (a is a cokernel of b).

In a category meeting Axiom 1, every morphism α admits a canonical decomposition $\alpha = (\operatorname{im}\alpha)\overline{\alpha}(\operatorname{coim}\alpha) = (\operatorname{im}\alpha)\widetilde{\alpha}$, where $\operatorname{im}\alpha = \ker\operatorname{coker}\alpha$, $\operatorname{coim}\alpha = \operatorname{coker}\ker\alpha$. Two canonical decompositions of the same morphism are obviously naturally isomorphic. A morphism α is called *strict* if $\overline{\alpha}$ is an isomorphism.

We use the following notations of [**12**]:
O_c is the class of all strict morphisms,
M is the class of all monomorphisms,
M_c is the class of all strict monomorphisms (= kernels),
P is the class of all epimorphisms,
P_c is the class of all strict epimorphisms (= cokernels).

LEMMA 2.1 ([**2, 3, 12, 22**]). *The following assertions hold in an additive category meeting Axiom* 1:

(1) $\ker\alpha \in M_c$ *and* $\operatorname{coker}\alpha \in P_c$ *for every* α;

(2) $\alpha \in M_c \iff \alpha = \operatorname{im}\alpha$, $\alpha \in P_c \iff \alpha = \operatorname{coim}\alpha$;

(3) *a morphism* α *is strict if and only if it is representable in the form* $\alpha = \alpha_1\alpha_0$ *with* $\alpha_0 \in P_c$, $\alpha_1 \in M_c$; *in every such representation,* $\alpha_0 = \operatorname{coim}\alpha$ *and* $\alpha_1 = \operatorname{im}\alpha$;

(4) *if some commutative square*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & D \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow{\beta} & B \end{array}$$

is a pullback then $\ker f = \alpha(\ker g)$ *and* $f = \ker\xi$ *implies* $g = \ker(\xi\beta)$; *in particular,* $f \in M \Longrightarrow g \in M$ *and* $f \in M_c \Longrightarrow g \in M_c$. *Dually, if the square is a pushout, then* $\operatorname{coker} g = (\operatorname{coker} f)\beta$ *and* $g = \operatorname{coker}\zeta$ *implies* $f = \operatorname{coker}(\alpha\zeta)$; *in particular,* $g \in P \Longrightarrow f \in P$ *and* $g \in P_c \Longrightarrow f \in P_c$.

An additive category meeting Axiom 1 is abelian if and only if $\overline{\alpha}$ is an isomorphism for every α. Consider the following axiom.

AXIOM 2. For every morphism α, $\overline{\alpha}$ is a bimorphism, i.e., a monomorphism and an epimorphism.

We write $\alpha\|\beta$ if the sequence $\cdot \xrightarrow{\alpha} \cdot \xrightarrow{\beta} \cdot$ is exact, that is, $\operatorname{im}\alpha = \ker\beta$ (which, in a category meeting Axioms 1 and 2, is equivalent to $\operatorname{coker}\alpha = \operatorname{coim}\beta$). The special case of this where $\alpha = \ker\beta$ and $\beta = \operatorname{coker}\alpha$ is written as $\alpha|\beta$.

LEMMA 2.2 ([**10**]). *The following assertions hold in an additive category satisfying Axioms* 1 *and* 2:

(1) *if* $gf \in M_c$ *then* $f \in M_c$; *if* $gf \in P_c$ *then* $g \in P_c$;

(2) *if* $f, g \in M_c$ *and* fg *is defined then* $fg \in M_c$*, if* $f, g \in P_c$ *and* fg *is defined then* $fg \in P_c$;

(3) *if* $fg \in O_c$ *and* $f \in M$ *then* $g \in O_c$*, if* $fg \in O_c$ *and* $g \in P$ *then* $f \in O_c$.

It is well known (see, for example, [**18**]), that every abelian category satisfies the following two axioms dual to one another.

AXIOM 3. If (1) is a pullback then $f \in P_c \Longrightarrow g \in P_c$.

AXIOM 4. If (1) is a pushout then $g \in M_c \Longrightarrow f \in M_c$.

An additive category satisfying Axioms 1, 3, and 4, is called *quasi-abelian*. Such categories are also known as (*Raĭkov*)*-semiabelian* (the original name, proposed by Raĭkov in [**22**] and used in the Russian tradition; now, however, the term *semi-abelian category* is involved in a quite different context [**8**]) or *almost abelian* [**24**]. As follows from Theorem 1 of [**12**], each quasi-abelian category meets Axiom 2.

Given an arbitrary commutative square (1.1), denote by $\widehat{g} : \operatorname{Ker}\alpha \longrightarrow \operatorname{Ker}\beta$ the morphism defined by the equality $g(\ker\alpha) = (\ker\beta)\widehat{g}$ and by $\widehat{f} : \operatorname{Coker}\alpha \longrightarrow \operatorname{Coker}\beta$ the morphism defined by the condition $\widehat{f}(\operatorname{coker}\alpha) = (\operatorname{coker}\beta)f$.

From now on, unless otherwise specified, the ambient category $\mathcal{A}$ is assumed quasi-abelian.

Lemmas 5 and 6 of [**10**] yield the following assertion.

LEMMA 2.3 ([**10**]). *Suppose that square* (1.1) *is a pullback. If* $\beta \in O_c$ *then* $\alpha \in O_c$ *and* $\widehat{f} \in M$.

Dually, if (1.1) *is a pushout and* $\alpha \in O_c$ *then* $\beta \in O_c$ *and* $\widehat{g} \in P$.

LEMMA 2.4 (The Composition Lemma). *Suppose that the composition* gf *of two morphisms* f *and* g *is defined. Then there exists a semiexact sequence*

$$(2.1)\quad 0 \longrightarrow \operatorname{Ker} f \xrightarrow{\varphi} \operatorname{Ker}(gf) \xrightarrow{\psi} \operatorname{Ker} g \xrightarrow{\chi} \operatorname{Coker} f \xrightarrow{\lambda} \operatorname{Coker}(gf) \xrightarrow{\omega} \operatorname{Coker} g \longrightarrow 0$$

which is exact at $\operatorname{Ker} f$*,* $\operatorname{Ker}(gf)$*,* $\operatorname{Coker}(gf)$*, and* $\operatorname{Coker} g$*; moreover,* φ *and* ω *are strict. Furthermore, if* $f \in O_c$ *then* (2.1) *is exact at* $\operatorname{Ker} g$ *and* $\psi \in O_c$*; if* $g \in O_c$ *then* (2.1) *is exact at* $\operatorname{Coker} f$ *and* $\lambda \in O_c$.

PROOF. As in an abelian category, we define φ, ψ, χ, λ, and ω by the equalities $\ker f = (\ker(gf))\varphi$, $f(\ker(gf)) = (\ker g)\psi$, $\chi = (\operatorname{coker} f)(\ker g)$, $(\operatorname{coker}(gf))g = \lambda(\operatorname{coker} g)$, and $\operatorname{coker} g = \omega(\operatorname{coker}(gf))$. Then it is standard (and easy) that sequence (2.1) thus obtained is semiexact, $\varphi = \ker\psi$, and $\omega = \operatorname{coker}\lambda$. Furthermore, it is easy to check that the square

$$\begin{array}{ccc} \operatorname{Ker}(gf) & \xrightarrow{\ker(gf)} & \cdot \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \operatorname{Ker} g & \xrightarrow{\ker g} & \cdot \end{array} \tag{2.2}$$

is a pullback.

Suppose that f is strict. Applying Lemma 2.3 to pullback (2.2), we see that $\psi \in O_c$ and the morphism l defined by the equality $l(\operatorname{coker}\psi) = (\operatorname{coker} f)(\ker g)$ $(= \chi)$ is monic. Thus, $\operatorname{im}\psi = \ker\chi$, which proves the exactness at $\operatorname{Ker} g$. By duality, we infer that $\lambda \in O_c$ and (2.1) is exact at $\operatorname{Coker} f$. □

3. Lambek Invariants

Given a commutative square (1.1), consider the pullback

$$\begin{array}{ccc} I & \xrightarrow{k} & \operatorname{Im} f \\ {\scriptstyle l}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{im} f} \\ \operatorname{Im}\beta & \xrightarrow{\operatorname{im}\beta} & B \end{array} \tag{3.1}$$

Easily, there are morphisms $k' : \operatorname{Im}(f\alpha) \longrightarrow \operatorname{Im} f$ and $l' : \operatorname{Im}(f\alpha) \longrightarrow \operatorname{Im}\beta$ with $\operatorname{im}(f\alpha) = (\operatorname{im} f)k' = (\operatorname{im}\beta)l'$. Since (3.1) is a pullback, there is a unique morphism $\rho : \operatorname{Im}(f\alpha) \longrightarrow I$ such that $k' = k\rho$ and $l' = l\rho$. We put $\operatorname{Im} S = \operatorname{Coker}\rho$. If we denote by Φ the epimorphism $\widetilde{f\alpha}$ then, obviously, $\operatorname{Im} S = \operatorname{Coker}(\rho\Phi)$.

Now, let $\mu : \operatorname{Ker} g \longrightarrow \operatorname{Ker}(f\alpha)$ and $\nu : \operatorname{Ker}\alpha \longrightarrow \operatorname{Ker}(f\alpha)$ be the natural inclusions. They form a morphism $\langle\mu,\nu\rangle : \operatorname{Ker} g \oplus \operatorname{Ker}\alpha \longrightarrow \operatorname{Ker}(f\alpha)$. We put $\operatorname{Ker} S = \operatorname{Coker}\langle\mu,\nu\rangle$. Alternatively, $\operatorname{Ker} S$ can be described as follows (see, for example, [**16**]). Consider the pushout

$$\begin{array}{ccc} C & \xrightarrow{\operatorname{coim}\alpha} & \operatorname{Coim}\alpha \\ {\scriptstyle \operatorname{coim} g}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ \operatorname{Coim} g & \xrightarrow{i} & J \end{array}$$

There is a unique morphism $\sigma : L \longrightarrow B$ such that $\sigma j = f(\operatorname{im}\alpha)\bar{\alpha}$ and $\sigma i = \beta(\operatorname{im} g)\bar{g}$. Then $\operatorname{Ker} S$ is naturally isomorphic with $\operatorname{Ker}\sigma$. Thus, $\operatorname{Im} S$ and $\operatorname{Ker} S$ are dual notions.

In what follows, we endow all the morphisms and objects introduced above for a commutative square S with the subscript S when it becomes necessary to distinguish the corresponding morphisms of different squares.

The condition $\operatorname{Im} S = 0$ $(\operatorname{Ker} S = 0)$ is fulfilled for an important class of pullbacks in a quasi-abelian category. Namely, the following assertion holds.

THEOREM 3.1. *Suppose that square* (1.1) *is a pullback with β and f strict. Then* $\operatorname{Im} S = 0$. *If* (1.1) *is a pushout with α and g strict then* $\operatorname{Ker} S = 0$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
F & \xrightarrow{v_1} & \cdot & \xrightarrow{w_1} & D \\
{\scriptstyle v_2}\downarrow & & {\scriptstyle v_0}\downarrow & & \downarrow{\scriptstyle \operatorname{coim} f} \\
\cdot & \longrightarrow & I & \xrightarrow{k} & \operatorname{Im} f \\
{\scriptstyle w_2}\downarrow & & {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle \operatorname{im} f} \\
A & \xrightarrow{\operatorname{coim}\beta} & \operatorname{Im}\beta & \xrightarrow{\operatorname{im}\beta} & B,
\end{array}$$

where all the four squares are pullbacks. Then the "resulting" square is a pullback, too (see, for example, [**3**], Proposition 2.10). Thus, up to an isomorphism, we have $C = F$, $w_1v_1 = \alpha$, and $w_2v_2 = g$. Since $w_1, w_2 \in M_c$ and $v_1, v_2 \in P_c$, by Lemma 2.1(3) it follows that $w_1 = \operatorname{im}\alpha$, $v_1 = \operatorname{coim}\alpha$, $w_2 = \operatorname{im} g$, and $v_2 = \operatorname{coim} g$. Therefore, $\operatorname{im}(f\alpha) = \operatorname{im}((\operatorname{im} f)kv_0v_1) = (\operatorname{im} f)k$, and hence $I = \operatorname{Im}(f\alpha)$, which implies $\operatorname{Im} S = 0$.

The second assertion is proved by duality. □

Remark 3.1. By Lemma 2.3, if square (1.1) is a pullback with $\beta \in O_c$ ($f \in O_c$) then $\alpha \in O_c$ ($g \in O_c$). This means that Theorem 3.1 applies to "strict" pullbacks. However, it fails for "nonstrict" pullbacks, which is demonstrated by the following example. Consider the category $\mathcal{B}an$ of Banach spaces and bounded linear operators. Let A and B be infinite-dimensional Banach spaces and let $\beta : A \longrightarrow B$ be a linear operator with dense range $R(\beta) \neq B$ (and so $\beta \notin O_c$!). Put $D = \mathbb{R}$ and suppose that $f : D \longrightarrow B$ is injective and $R(f) \cap R(\beta) = 0$. Form a pullback $f\alpha = \beta g$. For a morphism $L : X \longrightarrow Y$ in $\mathcal{B}an$, $\operatorname{Im} L$ is the closure $\overline{R(L)}$ of its range $R(L)$. It is easy to see that $\alpha = 0$ and hence $\operatorname{Im}(f\alpha) = 0$. However, in this case, $I = \overline{R(\beta)} \cap \overline{R(f)} \cong \mathbb{R} \neq 0$. Thus, $\operatorname{Im} S \cong \mathbb{R}$.

Remark 3.2. The class of commutative squares S with $\operatorname{Im} S = 0$ ($\operatorname{Ker} S = 0$) is not reduced to "strict" pullbacks (pushouts). As was observed by Hilton (see [**6**], Proposition 2.4) and is easily checked, each composition $h = gf$ yields two commutative squares $\Delta' :\ h(\mathrm{id}) = gf$ and $\Delta'' :\ (\mathrm{id})h = gf$ such that $\operatorname{Im}\Delta' = 0$ and $\operatorname{Ker}\Delta'' = 0$. Obviously, Δ' is a pullback if and only if g is monic (similarly, Δ'' is a pushout if and only if f is epic). Hence, a commutative square S need not be a pullback (pushout) to have $\operatorname{Im} S = 0$ ($\operatorname{Ker} S = 0$).

As we noted in the introduction, for a sequence of the form (1.2) with exact rows, $\operatorname{Ker} S$ and $\operatorname{Im} T$ are known to be naturally isomorphic (see [**14**] or [**16**]) in a Buchsbaum-exact category. For this to hold in a quasi-abelian category, one must have $\operatorname{Im} b = \operatorname{Coim} b$, that is, b must be strict. On the same assumption, we can use Nomura's construction of $\Lambda : \operatorname{Im} S \longrightarrow \operatorname{Ker} T$ for a diagram of the form (1.2) with semiexact rows. Recall that Λ is characterized by the equality $(\ker\sigma_T)\Lambda(\operatorname{coker}\rho_S) = i_Tk_S$ [**16**].

When the rows in (1.2) are exact, we can still construct a canonical morphism $\zeta : \operatorname{Ker} T \longrightarrow \operatorname{Im} S$. Of course, $\zeta = \Lambda^{-1}$ if Λ exists. Namely, we have

Theorem 3.2. *Suppose that in* (1.2) *the rows are exact. Then there exist unique morphisms* $\xi : \mathrm{Ker}(g'b) \longrightarrow I_S$ *and* $\zeta : \mathrm{Ker}\, T \longrightarrow \mathrm{Im}\, S$ *such that*

$$(\mathrm{coker}\, \rho_s)\xi = \zeta\, \mathrm{coker}\langle \mu_T, \nu_T\rangle.$$

Proof. Obviously, $g'b(\ker(g'b)) = 0$, which implies that there exists a unique morphism y with $b\ker(g'b)) = (\ker g')y_0 = (\mathrm{im}\, f')y_0$. Since (3.1) is a pullback, there exists a unique morphism $\xi : \mathrm{Ker}(g'b) \longrightarrow \mathrm{Im}\, S$ such that $\tilde{b}(\ker(g'b)) = k_S\xi$ and $y = l_S\xi$. We have

$$k_S\xi\mu_T = \tilde{b}(\ker(g'b))\mu_T = \tilde{b}(\ker b) = 0,$$

whence $\xi\mu_T = 0$ because k_S is monic. Now, denote by $\gamma = \gamma_S$ the unique morphism for which $\mathrm{im}(bf)\gamma = b(\mathrm{im}\, f)$ $(= b(\ker g)$ by the exactness of the upper row in (1.2)). We infer

$$\begin{aligned}(\mathrm{im}\, b)k_S\rho_S\gamma_S\tilde{f} &= (\mathrm{im}(bf))\gamma_S\tilde{f} = b(\mathrm{im}\, f)\tilde{f} = bf \\ &= (\mathrm{im}\, b)\tilde{b}(\ker(g'b))\nu_T\tilde{f} = (\mathrm{im}\, b)k_S\xi\nu_T\tilde{f}.\end{aligned}$$

Since $(\mathrm{im}\,\varphi)k \in M$ and $\tilde{f} \in P$, it follows that $\xi\nu_T = \rho_S\gamma_S$. Hence $(\mathrm{coker}\,\rho_S)\xi\nu_T = (\mathrm{coker}\,\rho_S)\xi\langle\mu_T, \nu_T\rangle = 0$. Therefore, there exists a unique morphism $\zeta : \mathrm{Coker}\langle\mu, \nu\rangle \longrightarrow \mathrm{Coker}\,\rho_S$ such that

$$(\mathrm{coker}\,\rho_S)\xi = \zeta\,\mathrm{coker}\langle\mu_T, \nu_T\rangle.$$

□

As a corollary to Theorem 3.2, we obtain Lambek's isomorphism, established for Buchsbaum-exact categories in [**13, 14, 16**], which, in our case, holds under the extra assumption that $b \in O_c$. Note that, in view of the exactness properties of the Ker-Coker-sequence in a quasi-abelian category proved in [**10**], Nomura's proof of Lambek's isomorphism in [**16**] is carried over to our situation literally. However, here we prefer to show how ζ becomes an isomorphism if b is strict.

Corollary 3.1. *If, under the conditions of Theorem* 3.2, $b \in O_c$ *then* ζ *is an isomorphism.*

Proof. First, observe that the square

$$\begin{array}{ccc} \mathrm{Ker}(g'b) & \xrightarrow{\ker(g'b)} & B \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\tilde{b}} \\ I & \xrightarrow{k_S} & \mathrm{Im}\, b \end{array} \tag{3.2}$$

is a pullback.

Indeed, suppose that morphisms x_1 and x_2 are such that $k_Sx_1 = \tilde{b}x_2$. Then

$$g'bx_2 = g'(\mathrm{im}\, b)\tilde{b}x_2 = g'(\mathrm{im}\, b)k_S\xi = g'(\mathrm{im}\, f')l_S\xi = g'(\ker g')l_S\xi = 0.$$

Therefore, there exists a unique morphism x with $x_2 = (\ker(g'b))x$. We now prove that $x_1 = \xi x$. We have

$$(\operatorname{im} b)k_S\xi x = (\operatorname{im} f')l_S\xi x = (\ker g')l_S\xi x = b(\ker(g'b))x$$
$$= (\operatorname{im} b)\tilde{b}(\ker(g'b))x = (\operatorname{im} b)\tilde{b}x_2 = (\operatorname{im} b)k_S x_1,$$

from which, by the fact that $(\operatorname{im} b)k_S$ is monic, we see that $\xi x = x_1$. Thus, we have demonstrated that (3.2) is a pullback.

Since $\ker b = \ker \tilde{b} = (\ker(g'b))\mu_T$, $b \in O_c$, and (3.2) is a pullback, from Lemma 2.1(4) and Axiom 3 it follows that $\mu_T|\xi$. Obviously, we have $(\operatorname{coker}\langle\mu_T, \nu_T\rangle)\mu_T = 0$, and so there exists a unique morphism $\tau : I \longrightarrow \operatorname{Coker}\langle\mu_T, \nu_T\rangle$ such that $\operatorname{coker}\langle\mu_T, \nu_T\rangle = \tau\xi$. We have $\zeta\tau\xi = (\operatorname{coker}\rho_S)\xi$, and the relation $\xi \in P_c$ yields $\zeta\tau = \operatorname{coker}\rho_S$. Furthermore,

$$\tau\rho_S\gamma_S = \tau\xi\nu_T = (\operatorname{coker}\langle\mu_T, \nu_T\rangle)\nu_T = 0. \tag{3.3}$$

Since $\gamma_S\tilde{f} = \Phi$, it follows that γ_S is epic and so (3.3) implies that $\tau\rho_S = 0$. Thus there is a unique morphism $\Lambda_0 : \operatorname{Coker}\rho \longrightarrow \operatorname{Coker}\langle\mu_T, \nu_T\rangle$ with the property $\tau = \Lambda_0(\operatorname{coker}\rho)$. Easily, $\zeta\Lambda_0$ and $\Lambda_0\zeta$ are identities and, therefore, ζ and Λ_0 are mutually inverse isomorphisms. □

It can be proved that, up to the identification $\operatorname{Ker} T \cong \operatorname{Ker}\sigma_T$, Λ_0 is Nomura's morphism Λ.

We now pass to the more general case of a commutative diagram of the form (1.2) with semiexact rows.

In the case of a Buchsbaum-exact category, Nomura constructed exact sequence (1.3). However, an analysis of the proof of the exactness of (1.3) in [**16**] (based on the Composition Lemma, cf. Lemma 2.4) shows that, in the quasi-abelian case, many morphisms must be assumed strict so that all morphisms in (1.3) can be defined. We prove the following quasi-abelian version of Corollary A_2 of [**16**].

Theorem 3.3. *Suppose that in diagram* (1.2) *the rows are semiexact. The following asserions hold.*

(1) *If the sequence $A' \to B' \to C'$ is exact and $b \in M_c$ then there exists a canonical morphism $\theta : H(A \to B \to C) \to \operatorname{Im} S$ such that the sequence*

$$0 \longrightarrow H(A \to B \to C) \xrightarrow{\theta} \operatorname{Im} S \xrightarrow{\Lambda} \operatorname{Ker} T \longrightarrow 0$$

is exact.

(2) *If the sequence $A \to B \to C$ is exact and $b \in P_c$ then there exists a canonical morphism $\varkappa : \operatorname{Ker} T \to H(A' \to B' \to C')$ such that the sequence*

$$0 \longrightarrow \operatorname{Im} S \xrightarrow{\Lambda} \operatorname{Ker} T \xrightarrow{\varkappa} H(A' \to B' \to C') \longrightarrow 0$$

is exact.

Proof. We prove only item (1) because item (2) is obtained from it by duality.

By definition, the homology object $H(A \to B \to C)$ is the cokernel of the unique morphism ε such that $\operatorname{im} f = (\ker g)\varepsilon$. Consequently, $(\operatorname{coker}\rho_S)\xi\nu_T\varepsilon = 0$ and, therefore, there exists a unique morphism θ with $(\operatorname{coker}\rho_S)\xi\nu_T = \theta(\operatorname{coker}\varepsilon)$. Repeating the argument of the proof of Theorem 3.2 almost literally, we see that $\rho_S\gamma_S = \xi\nu_T\varepsilon$.

Furthermore, since $b(\operatorname{im} f) = (\operatorname{im}(bf))\gamma_S$, b is a kernel, and the proof of Corollary 3.1 implies that γ_S is epic, it follows that γ_S is in fact an isomorphism. In addition, ξ is an isomorphism, too. Indeed, as above, ξ is a part of pullback (3.2), which implies that $\xi \in P_c$ and $(\ker(g'b))(\ker\xi) = \ker\tilde{b} = (\ker(g'b))\mu_T = 0$. Thus $\mu_T = 0$ and hence ξ is in fact an isomorphism. Thus we may write $\rho_S = \nu_T\varepsilon$. Since we thus obtain a pullback $\rho_S\,\mathrm{id} = \nu_T\varepsilon$ and $\nu_T \in O_c$, the morphism of the cokernels $\theta : \operatorname{Coker}\varepsilon \longrightarrow \operatorname{Coker}\rho_S$ is monic. Thus we see the exactness at $H(A \to B \to C)$.

Furthermore, since

$$\Lambda\theta(\operatorname{coker}\varepsilon) = \Lambda(\operatorname{coker}\rho_S)\xi\nu_T = (\operatorname{coker}\langle\mu_t,\nu_T\rangle)\nu_T = 0,$$

we infer $\Lambda\theta = 0$. Now, take a morphism y with $y\theta = 0$. Then $y(\operatorname{coker}\rho_S)\nu = y\theta(\operatorname{coker}\varepsilon) = 0$ and, obviously, $y(\operatorname{coker}\rho_S)\mu_T = 0$. Hence, there exists a unique morphism v with $y(\operatorname{coker}\rho_S) = v(\operatorname{coker}\langle\mu_T,\nu_T\rangle) = v\Lambda(\operatorname{coker}\rho_S)$. Since $\operatorname{coker}\rho_S$ is epic, $y = v\Lambda$. Thus, $\Lambda = \operatorname{coker}\theta$ and so we have the exactness at $\operatorname{Im} S$. $\square$

We now prove another assertion about a diagram of commutative squares (cf. Proposition 2.7 in [**6**]).

THEOREM 3.4. *Suppose that, in a commutative diagram*

$$\begin{array}{ccccc} A_1 & \xrightarrow{\theta_1} & B_1 & \xrightarrow{\theta_2} & C_1 \\ {\scriptstyle\alpha_1}\downarrow & S & \downarrow{\scriptstyle\beta_1} & T & \downarrow{\scriptstyle\gamma_1} \\ A_2 & \xrightarrow{\varphi_1} & B_2 & \xrightarrow{\varphi_2} & C_2 \\ {\scriptstyle\alpha_2}\downarrow & U & \downarrow{\scriptstyle\beta_2} & V & \downarrow{\scriptstyle\gamma_2} \\ A_3 & \xrightarrow{\psi_1} & B_3 & \xrightarrow{\psi_2} & C_3 \end{array}, \tag{3.4}$$

the first column is exact at A_2, the third, at C_2, and the second row is exact at B_2, $\operatorname{Im} T = 0$, $\operatorname{Ker} U = 0$, $\beta_2\beta_1 = 0$, $\varphi_1 \in O_c$, *and* $\varphi_2\beta_1 \in O_c$. *Then the second column is exact at B_2.*

PROOF. Take a morphism $x : X \longrightarrow B_2$ such that $\beta_2 x = 0$. We may assume that $x = \operatorname{im} x$. We have $\gamma_2\varphi_2 x = 0$; therefore, there exists a unique morphism y such that $\varphi_2 x = (\operatorname{im}\gamma_1)y$. Since $(\operatorname{im}\varphi_2)\tilde{\varphi}_2 x = (\operatorname{im}\gamma_1)y$ and $\operatorname{Im} T = 0$, there is a unique morphism $\xi : X \longrightarrow \operatorname{Im}(\varphi_2\beta_1)$ with the properties $\tilde{\varphi}_2 x = l_T\xi$ and $y = k_T\xi$. Thus, $\varphi_2 x = (\operatorname{im}\varphi_2)l\xi = \operatorname{im}(\varphi_2\beta_1)\xi$. Define ω by the equality $\varphi_2\beta_1 = \operatorname{im}(\varphi_2\beta_1)\omega$. Then $\omega \in P_c$. Consider a pullback $\xi\omega_0 = \omega\xi_0$. We have $\operatorname{im}(\varphi_2\beta_1)\xi\omega_0 = \operatorname{im}(\varphi_2\beta_1)\omega\xi_0 = \varphi_2\beta_1\xi_0$. Thus, $\varphi_2(x\omega_0 - \beta_1\xi_0) = 0$, from which we deduce the existence of a unique morphism ξ_1 such that $x\omega_0 - \beta_1\xi_0 = (\ker\varphi_2)\xi_1 = (\operatorname{im}\varphi_1)\xi_1$. Let $\xi_0 p_0 = \tilde{\varphi}_1\xi_1$ be a pullback. Then

$$0 = \beta_2 x\omega_0 = \beta_2(\operatorname{im}\varphi_1)\xi_0 p_0 = \beta_2(\operatorname{im}\varphi_1)\tilde{\varphi}_1\xi_1 = \beta_2\varphi_1\xi_1.$$

Since $\operatorname{Ker} U = 0$, it follows that $\operatorname{Ker}(\beta_2\varphi_1) \cong \operatorname{Ker}\varphi_1 \oplus \operatorname{Ker}\alpha_2$. Consequently, $\xi_1 = (\ker\varphi_1)t_1 + (\ker\alpha_2)t_2 = (\ker\varphi_1)t_1 + (\operatorname{im}\alpha_1)t_2$ for some t_1 and t_2. Furthermore, there exists a unique morphism u with $\varphi_1(\operatorname{im}\alpha_1) = (\operatorname{im}\beta_1)u$. We infer

$$x\omega_0 p_0 = \beta_1\xi_0 p_0 + \varphi_1(\operatorname{im}\alpha_1)t_2 = \beta_1\xi_0 p_0 + (\operatorname{im}\beta_1)ut_2 = (\operatorname{im}\beta_1)(\tilde{\beta}_1\xi_0 p_0 + ut_2).$$

Thus, $x\omega_0 p_0 = (\operatorname{im}\beta_1)v$, i. e., $x\omega_0 p_0 = (\operatorname{im}\beta_1)(\operatorname{im} v)\bar{v}(\operatorname{coim} v)$. The hypothesis implies that $\omega_0 p_0 \in P_c$. Therefore, $x = (\operatorname{im}\beta_1)(\operatorname{im} v)$, which means that $\operatorname{im}\beta_1 = \ker\beta_2$. □

For abelian categories, Theorem 3.4 was proved by Hilton (see [**6**], Proposition 2.7) and served as a key ingredient in the proof of the main theorem in [**6**] on the exactness of a system of interlocking exact sequences. In the quasi-abelian case, we have to add some strictness conditions to Hilton's Proposition 2.7. Unfortunately, applying Theorem 3.4 to interlocking sequences (and thus to spectral sequences) is possible only if we assume all the morphisms strict. We dealt with spectral sequences by considering exact couples in quasi-abelian categories in a separate paper [**9**].

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