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Gain of regularity for a nonlinear dispersive equation

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Abstract. In this paper we study the gain of regularity of solutions of a dispersive evolution nonlinear equation. We consider the equation

$$(1) = \begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[\, f(u)\,] = \epsilon \, \frac{\partial}{\partial x}[\, g(\frac{\partial u}{\partial x})\,] - \delta \, \frac{\partial^3 u}{\partial x^3} \\ u(x,\, 0) = \varphi(x) \end{cases}$$

where $x \in \mathbb{R},\, t \in [0,\, T]$ and T is an arbitrary positive time. The flux $f = f(u)$ and the (degenerate) viscosity $g = g(\lambda)$ are smooth functions satisfying certain assumptions to be listed below. It is shown under certain additional conditions on f that C^∞ - solutions $u(x,\, t)$ are obtained for all $t > 0$ if the initial data $u(x,\, 0) = \varphi(x)$ decays faster than polinomially on $\mathbb{R}^+ = \{x \in \mathbb{R}\,;\, x > 0\}$ and it has certain initial Sobolev regularity.

1. Introduction

In 1976, J. C. Saut and R. Temam [**23**] noted that a solution u of an equation of Korteweg-de Vries type cannot gain or lose regularity. They showed that if $u(x,\, 0) = \varphi(x) \in H^s(\mathbb{R})$ for $s \geqslant 2$, then $u(\,\cdot\,,\, t) \in H^s(\mathbb{R})$ for all $t > 0$. The same results were independently obtained making use of different methods by J. Bona and R. Scott [**2**]. For the Korteweg - de Vries (KdV) equation on the line, and motivated by the work of A. Cohen [**6**], T. Kato [**16**] showed that if $u(x,\, 0) = \varphi(x) \in L^2_b \equiv H^2(\mathbb{R}) \bigcap L^2(e^{bx}dx)\,(b > 0)$, then the solution $u(x,\, t)$ of the KdV equation becomes C^∞ for all $t > 0$. The main ingredient in the proof was the fact that formally the semi-group $S(t) = e^{-t\,\partial_x^3}$ in L^2_b is equivalent to $S_b(t) = e^{-t\,(\partial_x - b)^3}$ in L^2, when $t > 0$. One would be inclined to believe that this was a special property of the KdV equation. This is not, however, the case. The effect is due to the dispersive nature of the linear part of the equation. S. N. Kruzkov and A. V. Faminskii [**20**] proved that for $u(x,\, 0) = \varphi(x) \in L^2$, such that $x^\alpha \varphi(x) \in L^2((0,\, +\infty))$, the weak solution of the KdV

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equation had l-continuous space derivatives for all $t > 0$ if $l < 2\,\alpha$. The proof of this result was based on the asymptotic behavior of the Airy function and its derivatives, and on the smooth effect of the KdV equation found in [**16, 20**]. Corresponding work for special nonlinear Schrödinger equations were done by Hayashi et al. [**12, 13**] and G. Ponce [**22**]. While the proof of T. Kato turns out to be dependent on special a priori estimates, some of its mystery has been resolved by results of local gain of finite regularity for various other linear and nonlinear dispersive equations due to P. Constantin and J. C. Saut [**10**], P. Sjolin [**24**], J. Ginibre and G. Velo [**11**], and others. However, all of them require growth conditions on the nonlinear term.

All physically significant dispersive equations and systems are known to have linear parts displaying this local smooth property. We mention only a few: the KdV, Benjamin-Ono, intermediate long wave, various Boussinesq, and Schrodinger equations. Continuing with the idea of W. Craig, T. Kappeler and W. Strauss [**9**] we study smoothness properties of solutions of some evolution dispersive nonlinear equations. We consider the nonlinear dispersive equation

$$(1) = \begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[\,f(u)\,] = \epsilon\,\frac{\partial}{\partial x}[\,g(\frac{\partial u}{\partial x})\,] - \delta\,\frac{\partial^3 u}{\partial x^3} \\ u(x,\,0) = \varphi(x) \end{cases}$$

with $x \in \mathbb{R}$, $t \in [0,\,T]$ and T is an arbitrary positive time. The flux $f = f(u)$ and the (degenerate) viscosity $g = g(\lambda)$ are given by smooth functions satisfying certain assumptions that we list below. We give a formal proof of our gain in a regularity theorem for the nonlinear equation (1). The results of Section 3 were proved in O. Vera [**27**]. In Section 4 we state our main results on the gain of regularity for (1). We prove the following principal theorem.

THEOREM 1.1. (Main Theorem) *Let $T > 0$ and $u(x,\,t)$ be a solution of (1) in the region $\mathbb{R} \times [0,\,T]$, such that $u \in L^\infty([0,\,T];\,H^3(W_{0\,L\,0}))$ for some $L \geqslant 2$ and all $\sigma > 0$. Then*

$$u \in L^\infty([0,\,T];\,H^{3+l}(W_{\sigma,\,L-l,\,l}))\bigcap L^2([0,\,T];\,H^{4+l}(W_{\sigma,\,L-l-1,\,l})),$$

for all $0 \leqslant l \leqslant L-1$.

2. Preliminaries

We consider the nonlinear dispersive equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\,[f(u)] = \epsilon\frac{\partial}{\partial x}\left[g\left(\frac{\partial u}{\partial x}\right)\right] - \delta\frac{\partial^3 u}{\partial x^3} \tag{2.1}$$

with $x \in \mathbb{R}$, $t \in [0,\,T]$ and T an arbitrary positive time. The flux $f = f(u)$ and the viscosity $g = g(\lambda)$ are given by smooth functions satisfying certain assumptions and $\epsilon,\,\delta > 0$.

Notation 1. We write $\partial = \frac{\partial}{\partial x}$; $\partial_t u = \frac{\partial u}{\partial t} = u_t$ and abbreviate $u_j = \partial^j u = \frac{\partial^j u}{\partial x^j}$; $\partial_j = \frac{\partial}{\partial u_j}$.

EXAMPLE 2.1. If $\partial u/\partial x = u_1$ then

$$\frac{\partial}{\partial x}\left[g\left(\frac{\partial u}{\partial x}\right)\right] = \frac{\partial}{\partial x}\,[\, g(u_1)\,] = \frac{\partial}{\partial u_1}\,[\, g(u_1)\,]\ \frac{\partial}{\partial x}[\, u_1\,] = \frac{\partial}{\partial u_1}\,[\, g(u_1)\,]\ u_2 = (\partial_1 g)\, u_2.$$

The assumptions on f are:

A.1 $f\colon \mathbb{R}^2 \times [0,\, T] \mapsto \mathbb{R}$ is C^∞ with respect to all its variables.

A.2 All derivatives of $f = f(u,\, x,\, t)$ are bounded for $x \in \mathbb{R}$, $t \in [0,\, T]$ and $u \in \mathbb{R}$ lies in a bounded set.

A.3 $x^N \partial_x^j f(0,\, x,\, t)$ is bounded for all $N \geqslant 0$, $j \geqslant 0$, and $x \in \mathbb{R}$, $t \in (0,\, T]$. Indeed, $\forall\, N \geqslant 0$, $\forall j \geqslant 0$, $x \in \mathbb{R}$, $t \in (0,\, T]$ there exists $c > 0$ such that $|x^N \partial_x^j f(0,\, x,\, t\,)| \leqslant c$.

The assumptions on g are as follows:

B.1 $g\colon \mathbb{R}^2 \times [0,\, T] \mapsto \mathbb{R}$ is C^∞ with respect to all its variables.

B.2 All the derivatives of $g(y,\, x,\, t)$ are bounded for $x \in \mathbb{R}$, $t \in [0,\, T]$ and y is from a bounded set.

B.3 $x^N \partial_x^j g(0,\, x,\, t)$ is bounded for all $N \geqslant 0$, $j \geqslant 0$ and $x \in \mathbb{R}, t \in (0,\, T]$.

B.4 There exists $c > 0$, such that $\partial_1 g(u_1,\, x,\, t) \geqslant c > 0$, for all $u_1 \in \mathbb{R}$; $x \in \mathbb{R}$ and $t \in [0,\, T]$.

LEMMA 2.1. *The assumptions A.1–A.3 imply that f has the form $f = u_0 f_0 + h \equiv u f_0 + h$, where $f_0 = f_0(u_0,\, x,\, t) \equiv f_0(u,\, x,\, t)$, and $h = h(x,\, t)$. f_0, and h are C^∞ and each of their derivatives is bounded for u bounded, $x \in \mathbb{R}$ and $t \in [0,\, T]$.*

PROOF. We define (the same for g)

$$f_0 = \begin{cases} \frac{f(u_0,\, x,\, t) - f(0,\, x,\, t)}{u_0} & \text{for } u_0 \neq 0 \\ \partial_0 f(0,\, x,\, t) & \text{for } u_0 = 0 \end{cases}$$

and $h(x,\, t) = f(0,\, x,\, t)$. □

DEFINITION 2.1. An evolution equation enjoys a *gain of regularity* if its solutions are smoother for all $t > 0$ than its initial data.

DEFINITION 2.2. A function $\xi(x,\, t)$ belongs to the weight class $W_{\sigma\, i\, k}$ if it is a positive C^∞ function on $\mathbb{R} \times [0,\, T]$, $\xi_x > 0$ and there exist constants c_j, $0 \leqslant j \leqslant 5$, such that

$$0 < c_1 \leqslant t^{-k} e^{-\sigma x} \xi(x,\, t) \leqslant c_2 \qquad \text{for} \quad x < -1, \quad 0 < t < T. \tag{2.2}$$

$$0 < c_3 \leqslant t^{-k} x^{-i} \xi(x,\, t) \leqslant c_4 \qquad \text{for} \quad x > 1, \quad 0 < t < T. \tag{2.3}$$

$$\big(t \mid \xi_t \mid + \mid \partial^j \xi \mid\big) / \xi \leqslant c_5 \quad \text{in} \quad \mathbb{R} \times [0,\, T],\ \forall\, j \in \mathbb{N}. \tag{2.4}$$

REMARK 2.1. We will always consider $\sigma \geqslant 0$, $i \geqslant 1$ and $k \geqslant 0$.

EXAMPLE 2.2. Let

$$\xi(x) = \begin{cases} 1 + e^{-1/x} & \text{for } x > 0, \\ 1 & \text{for } x \leqslant 0. \end{cases}$$

Then $\xi \in W_{0\, i\, 0}$.

DEFINITION 2.3. For a fixed $\xi \in W_{\sigma\, i\, k}$ we define the space (for a positive integer s)

$$H^s(W_{\sigma\, i\, k}) = \{v\colon \mathbb{R} \to \mathbb{R};\ \text{such that the distributional derivatives } \frac{\partial^j v}{\partial x^j}$$
$$\text{for } 0 \leqslant j \leqslant s \text{ satisfy } \|v\|^2 = \sum_{j=0}^{s} \int_{-\infty}^{+\infty} |\partial^j v(x)|^2 \xi(x,\, t)dx < +\infty\}.$$

REMARK 2.2. $H^s(W_{\sigma\, i\, k})$ depends on t (because $\xi = \xi(x,\, t)$).

LEMMA 2.2. *For $\xi \in W_{\sigma\, i\, 0}$ and $\sigma \geqslant 0,\, i \geqslant 0$ there exists a constant c, such that*

$$\sup_{x \in \mathbb{R}} |\, \xi u^2 \,| \leqslant c \int_{-\infty}^{+\infty} \left(|\, u\,|^2 + |\, \partial u\,|^2 \right) \xi dx$$

for $u \in H^1(W_{\sigma\, i\, 0})$.

PROOF. See Lemma 7.3 in [**9**]. □

DEFINITION 2.4. For a fixed $\xi \in W_{\sigma\, i\, k}$ we define the space

$$L^2([0,\, T];\, H^s(W_{\sigma\, i\, k})) = \{v = v(x,\, t),\, v(\,\cdot\,,\, t) \in H^s(W_{\sigma\, i\, k}),\ \text{such that}$$
$$|||\, v\,|||^2 = \int_0^T \|v(\,\cdot,\, t)\|^2 dt < +\infty\};$$
$$L^\infty([0,\, T];\, H^s(W_{\sigma\, i\, k})) = \{v = v(x,\, t),\, v(\,\cdot\,,\, t) \in H^s(W_{\sigma\, i\, k})\ \text{such that}$$
$$|||\, v\,|||_\infty = \operatorname{ess}\sup_{t \in [0,\, T]} \|v(\,\cdot,\, t)\| < +\infty\}.$$

REMARK 2.3. The usual Sobolev space is $H^s(\mathbb{R}) = H^s(W_{0\,0\,0})$ without a weight.

REMARK 2.4. We shall derive a priori estimates assuming that the solution is C^∞, bounded as $x \to -\infty$, and rapidly decreasing as $x \to +\infty$, together with all of its derivatives.

According to the notation 1, we obtain

$$u_t + \delta u_3 - \epsilon(\partial_1 g)u_2 + (\partial_0 f)u_1 = 0 \tag{2.5}$$

for equation (1). The equation is considered for $-\infty < x < +\infty,\, t \in [0,\, T]$ and T arbitrary positive.

We would like to construct a mapping $T : L^\infty([0,\, T];\, H^s(\mathbb{R})) \to L^\infty([0,\, T];\, H^s(\mathbb{R}))$ with the following property. Given $u^{(n)} = T(u^{(n-1)})$ and $\|u^{(n-1)}\|_s \leqslant c_0$ we have $\|u^{(n)}\|_s \leqslant c_0$, where s and $c_0 > 0$ are constants. In fact, this property tells us that $T : \mathbb{B}_{c_0}(0) \to \mathbb{B}_{c_0}(0)$ where $\mathbb{B}_{c_0}(0) = \{v(x,\, t);\, \|v(x,\, t)\|_s \leqslant c_0\}$ is a ball in the space $L^\infty([0,\, T];\, H^s(\mathbb{R}))$. To guarantee this property we will appeal to an a priori estimate which is the main object of this section.

By differentiating (2.5) twice we obtain

$$\begin{aligned} &\partial_t u_2 + \delta\, u_5 - \epsilon\, (\partial_1 g)\, u_4 + (\partial_0 f)\, u_3 - 2\epsilon\, \partial(\partial_1 g)\, u_3 \\ &\quad + 2\partial(\partial_0 f)\, u_2 - \epsilon\, \partial^2(\partial_1 g)\, u_2 + \partial^2(\partial_0 f)\, u_1 = 0. \end{aligned} \tag{2.6}$$

Let $u = \Lambda v$ where $\Lambda = (I - \partial^2)^{-1}$. Then, $\partial_t u_2 = -v_t + u_t$. By replacing in (2.1) we have

$$\begin{aligned} & -v_t + \delta\Lambda v_5 - \epsilon(\partial_1 g)\Lambda v_4 + (\partial_0 f)\Lambda v_3 - 2\epsilon\partial(\partial_1 g)\Lambda v_3 + 2\partial(\partial_0 f)\Lambda v_2 \\ & -\epsilon\partial^2(\partial_1 g)\Lambda v_2 + \partial^2(\partial_0 f)\Lambda v_1 - [\delta\Lambda v_3 - \epsilon(\partial_1 g)\Lambda v_2 + (\partial_0 f)\Lambda v_1] = 0 \end{aligned} \tag{2.7}$$

where $g = g(\Lambda v_1)$ and $f = f(\Lambda v)$.

The equation (2.7) is linearized by substituting a new variable w in each coefficient;

$$\begin{aligned} & -v_t + \delta\Lambda v_5 - \epsilon\partial_1 g(\Lambda w_1)\Lambda v_4 + \partial_0 f(\Lambda w)\Lambda v_3 - 2\,\epsilon\partial(\partial_1 g(\Lambda w_1))\Lambda v_3 \\ & + 2\partial(\partial_0 f(\Lambda w))\Lambda v_2 - \epsilon\partial^2(\partial_1 g(\Lambda w_1))\Lambda v_2 + \partial^2(\partial_0 f(\Lambda w))\Lambda v_1 \\ & -[\delta\Lambda v_3 - \epsilon\partial_1 g(\Lambda w_1)\Lambda v_2 - \partial_0 f(\Lambda w)\Lambda v_1] = 0 \end{aligned} \tag{2.8}$$

LEMMA 2.3. *Let $v, w \in C^k([0,\infty); H^N(\mathbb{R}))$ for all k, N, which satisfy (2.8). Let $\xi \geqslant c_1 > 0$. For each integer α there exist positive nondecreasing functions E, F and G such that for all $t \geqslant 0$*

$$\partial_t \int_{\mathbb{R}} \xi v_\alpha^2 dx \leqslant G(\|w\|_\lambda)\|v\|_\alpha^2 + E(\|w\|_\lambda)\|w\|_\alpha^2 + F(\|w\|_\alpha) \tag{2.9}$$

where $\|\cdot\|_\alpha$ is the norm in $H^\alpha(\mathbb{R})$ and $\lambda =$ max $\{1, \alpha\}$.

PROOF. See Lemma 3.1 in [**27**]. □

We define a sequence of approximations to equation (2.8) as

$$\begin{aligned} & -v_t^{(n)} + \delta\Lambda v_5^{(n)} - \epsilon(\partial_1 g)\Lambda v_4^{(n)} + (\partial_0 f)\Lambda v_3^{(n)} \\ & -2\epsilon\partial(\partial_1 g)\Lambda v_3^{(n)} - \delta\Lambda v_3^{(n)} + O(\Lambda v_2^{(n-1)}, \Lambda v_1^{(n-1)}, \ldots) = 0, \end{aligned} \tag{2.10}$$

where $g = g(\Lambda v_1^{(n-1)})$, $f = f(\Lambda v^{(n-1)})$, and where the initial condition is given by $v^{(n)}(x,0) = \varphi(x) - \partial^2\varphi(x)$. The first approximation is given by $v^{(0)}(x,0) = \varphi(x) - \partial^2\varphi(x)$. Equation (2.10) is a linear equation at each iteration which can be solved in any time interval in which the coefficients are defined. This equation has the form

$$\partial_t v = \delta\Lambda v_5 - \epsilon\Lambda v_4 + b^{(1)}\Lambda v_3 + b^{(0)} \tag{2.11}$$

LEMMA 2.4. *Given initial data in $\varphi \in H^\infty(\mathbb{R}) = \bigcap_{N\geqslant 0} H^N(\mathbb{R})$, there exists a unique solution of (2.11). The solution is defined in any time interval in which the coefficients are defined.*

PROOF. See [**26**]. □

THEOREM 2.1. (Uniqueness) *Let $\varphi \in H^3(\mathbb{R})$ and $0 < T < +\infty$. Assume that f satisfies A.1-A.3. and g satisfies B.1-B.4, then there is at most one solution $u \in L^\infty([0,T]; H^3(\mathbb{R}))$ of (2.5) with the initial data $u(x,0) = \varphi(x)$.*

PROOF. See Theorem 4.1 in [**27**]. □

We construct the mapping $T : L^\infty([0,T]; H^s(\mathbb{R})) \to L^\infty([0,T]; H^s(\mathbb{R}))$ by

$$\begin{aligned} u^{(0)} &= \varphi(x) \\ u^{(n)} &= T(u^{(n-1)}) \qquad n \geqslant 1 \end{aligned}$$

where $u^{(n-1)}$ replaces w in (2.8) and $u^{(n)}$ replaces v which is the solution of (2.8). By Lemma 2.4 there exists a unique $u^{(n)}$ in $C((0,+\infty); H^N(\mathbb{R}))$. The choice of c_0 and the use of the a priori estimate shows that $T: \mathbb{B}_{c_0}(0) \to \mathbb{B}_{c_0}(0)$, where $\mathbb{B}_{c_0}(0)$ is a bounded ball in $L^\infty([0,T]; H^s(\mathbb{R}))$.

THEOREM 2.2. (Local Existence) *Assume that f satisfies A.1 - A.4, and g satisfies B.1 - B.4. Let $N \geqslant 3$ be integer. If $\varphi \in H^N(\mathbb{R})$, then there is $T > 0$ and u, such that u is a strong solution of (2.5). $u \in L^\infty([0,T]; H^N(\mathbb{R}))$ with the initial data $u(x,0) = \varphi(x)$.*

PROOF. See Theorem 4.2 in [**27**]. □

COROLLARY 2.1. *Let $\varphi \in H^N(\mathbb{R})$ with $N \geqslant 3$, such that $\varphi^{(\gamma)} \to \varphi$ in $H^N(\mathbb{R})$. Let u and $u^{(\gamma)}$ be the corresponding unique solutions given by Theorems 2.1 and Theorem 2.2 in $L^\infty([0,T]; H^N(\mathbb{R}))$ with T depending only on $\sup_\gamma \|\varphi^{(\gamma)}\|_{H^3(\mathbb{R})}$. Then $u^{(\gamma)} \overset{*}{\rightharpoonup} u$ weakly in $L^\infty([0,T]; H^N(\mathbb{R}))$ and $u^{(\gamma)} \to u$ strongly in $L^2([0,T]; H^{N+1}(\mathbb{R}))$.*

THEOREM 2.3. (Persistence) *Let $i \geqslant 1$ and $L \geqslant 3$ be non-negative integers, $0 < T < +\infty$. Assume that u is the solution of (2.5) in $L^\infty([0,T]; H^3(\mathbb{R}))$ with the initial data $\varphi(x) = u(x,0) \in H^3(\mathbb{R})$. If $\varphi(x) \in H^L(W_{0\,i\,0})$ then*

$$u \in L^\infty([0,T]; H^3(\mathbb{R}) \bigcap H^L(W_{0\,i\,0})) \tag{2.12}$$

where σ is arbitrary, $\eta \in W_{\sigma,\, i-1,\, 0}$ for $i \geqslant 1$.

PROOF. Similar to Theorem 2.2. □

3. Main inequality

LEMMA 3.1. *Let u be a solution of the initial value problem (2.5). Then we have the following inequality*

$$\partial_t \int_{\mathbb{R}} \xi u_\alpha^2 dx + \int_{\mathbb{R}} \eta u_{\alpha+1}^2 dx + \int_{\mathbb{R}} \theta_\alpha u_\alpha^2 dx + \int_{\mathbb{R}} R_\alpha dx \leqslant 0, \tag{3.1}$$

with

$$\begin{aligned} \eta &= (3\delta + 2\epsilon c)\,\partial\xi, \\ \theta_\alpha &= -\xi_t - \delta\partial^3\xi - 2\epsilon\partial^2[(\partial_1 g)\xi] + \epsilon(\alpha+1)\partial[(\partial_1^2 g)\xi u_2] - \partial(\xi\partial_0 f) \\ &\quad + 2\epsilon \binom{\alpha+1}{2} [(\partial_1^2 g)u_3 + \partial(\partial_1^2 g)u_2] + 2(\alpha+1)(\partial_0^2 f)\xi u_1, \\ R_\alpha &= O(u_\alpha, \ldots). \end{aligned}$$

PROOF. Taking α-derivatives of the equation (2.5) (for $\alpha \geqslant 3$) regarding to $x \in \mathbb{R}$ we obtain

$$\begin{aligned} &\partial_t u_\alpha + \delta\, u_{\alpha+3} - \epsilon(\partial_1 g)u_{\alpha+2} - \epsilon(\alpha+1)(\partial_1^2 g)u_2 u_{\alpha+1} + (\partial_0 f)u_{\alpha+1} \\ &-\epsilon \binom{\alpha+1}{2} [(\partial_1^2 g)u_3 + \partial(\partial_1^2 g)u_2]u_\alpha \end{aligned} \tag{3.2}$$

$$+(\alpha+1)(\partial_0^2 f)u_1 u_\alpha + O(u_{\alpha-1},\, u_{\alpha-2},\, \ldots) = 0.$$

By multiplying (3.2) by $2\xi u_\alpha$ and integrating over $x \in \mathbb{R}$ we have

$$2\int_{\mathbb{R}} \xi u_\alpha \partial_t u_\alpha dx + 2\delta \int_{\mathbb{R}} \xi u_\alpha u_{\alpha+3} dx - 2\epsilon \int_{\mathbb{R}} \xi(\partial_1 g) u_\alpha u_{\alpha+2} dx$$

$$-2\epsilon(\alpha+1)\int_{\mathbb{R}} \xi(\partial_1^2 g) u_2 u_\alpha u_{\alpha+1} dx + 2\int_{\mathbb{R}} \xi(\partial_0 f) u_\alpha u_{\alpha+1} dx$$

$$-2\epsilon \begin{pmatrix} \alpha+1 \\ 2 \end{pmatrix} \int_{\mathbb{R}} [(\partial_1^2 g)u_3 + \partial(\partial_1^2 g)u_2]\xi u_\alpha^2 dx + 2(\alpha+1)\int_{\mathbb{R}} (\partial_0^2 f) u_1 u_\alpha^2 dx$$

$$+\int_{\mathbb{R}} 2\xi u_\alpha O(u_{\alpha-1},\, u_{\alpha-2},\, \ldots) dx = 0.$$

Integrating by parts leads us to

$$\partial_t \int_{\mathbb{R}} \xi u_\alpha^2 dx + \int_{\mathbb{R}} (3\delta + 2\epsilon(\partial_1 g))\, \partial\xi u_{\alpha+1}^2 dx + \int_{\mathbb{R}} \theta_\alpha u_\alpha^2 dx + \int_{\mathbb{R}} R_\alpha dx = 0$$

with θ_α and R_α, as above. Using B.4 we come to the *main inequality*

$$\partial_t \int_{\mathbb{R}} \xi u_\alpha^2 dx + \int_{\mathbb{R}} (3\delta + 2\epsilon c)\, \partial\xi u_{\alpha+1}^2 dx + \int_{\mathbb{R}} \theta_\alpha u_\alpha^2 dx + \int_{\mathbb{R}} R_\alpha dx \leqslant 0.$$

□

LEMMA 3.2. *If $\eta \in W_{\sigma\, i\, k}$ is an arbitrary weight function, then there exists $\xi \in W_{\sigma,\, i+1,\, k}$ which satisfies*

$$\eta = (3\delta + 2\epsilon c)\, \partial\xi. \tag{3.3}$$

PROOF. Indeed

$$\xi = \frac{1}{(3\delta + 2\epsilon c)} \int_{-\infty}^{x} \eta(y,\, t) dy. \tag{3.4}$$

□

LEMMA 3.3. *The expression R_α in the main inequality is a sum of terms of the form*

$$\xi \partial_0^{p_0} \partial_x^\gamma f u_{\nu_1} u_{\nu_2} \ldots u_{\nu_{p-1}} u_{\nu_p} u_\alpha \ \text{ and } \quad \xi \partial_1^{p_1} \partial_x^\gamma g u_{\mu_1} u_{\mu_2} \ldots u_{\mu_{q-1}} u_{\mu_q} u_\alpha, \tag{3.5}$$

where $1 \leqslant \nu_1 \leqslant \nu_2 \leqslant \ldots \leqslant \nu_p \leqslant \alpha$ and $1 \leqslant \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_q \leqslant \alpha$.

$$\begin{aligned} p &= p_0 \geqslant 1 \quad , \quad \gamma = \alpha + 1 \\ q &= q_1 \geqslant 1 \quad , \quad \gamma = \alpha + 1 \end{aligned} \tag{3.6}$$

$$\begin{aligned} \nu_1 + \nu_2 + \ldots + \nu_p &= \alpha + 1 \\ \mu_1 + \mu_2 + \ldots + \mu_q &= \alpha + q_1 + 1 \end{aligned} \tag{3.7}$$

$$\begin{aligned} p + \nu_{p-1} + \nu_p &\leqslant \alpha + 3 \\ q + \mu_{q-1} + \mu_q &\leqslant \alpha + 4 \end{aligned} \tag{3.8}$$

4. Main theorem

In this section we state and prove our main theorem which tells us that if the initial data $u(x,\,0)$ decays faster than polynomially on $\mathbb{R}^+ = \{x \in \mathbb{R}\,;\ x > 0\}$ and possesses certain initial Sobolev regularity, then the solution $u(x,\,t) \in C^\infty$ for all $t > 0$.

In the main theorem we assume $4 \leqslant \alpha \leqslant L+2$. For $\alpha \leqslant L+2$, we take an arbitrary

$$\eta \in W_{\sigma,\,L-\alpha+2,\,\alpha-3} \Longrightarrow \xi \in W_{\sigma,\,L-\alpha+3,\,\alpha-3} \tag{4.1}$$

LEMMA 4.1. (Estimate of Error Terms) *Let $4 \leqslant \alpha \leqslant L+2$ and the weight functions be chosen as in (4.1), then*

$$\left| \int_0^T \int_{\mathbb{R}} \left(\theta u_\alpha^2 + R\right) dxdt \right| \leqslant c \tag{4.2}$$

where c depends only on the norms of u taken in

$$L^\infty([0,\,T];\,H^\beta(W_{\sigma,\,L-\beta+2,\,\beta-3})) \bigcap L^2([0,\,T];\,H^{\beta+1}(W_{\sigma,\,L-\beta+1,\,\beta-3}))$$

for $3 \leqslant \beta \leqslant \alpha - 1$, and the norms of u taken in $L^\infty([0,\,T];\,H^3(W_{0\,L\,0}))$.

THEOREM 4.1. (Main Theorem) *Let $T > 0$ and $u(x,\,t)$ be a solution of (2.5) in the region $\mathbb{R} \times [0,\,T]$, such that*

$$u \in L^\infty([0,\,T];\,H^3(W_{0\,L\,0})) \tag{4.3}$$

for some $L \geqslant 2$ and all $\sigma > 0$. Then,

$$u \in L^\infty([0,\,T];\,H^{3+l}(W_{\sigma,\,L-l,\,l})) \bigcap L^2([0,\,T];\,H^{4+l}(W_{\sigma,\,L-l-1,\,l}))$$

for all $0 \leqslant l \leqslant L-1$.

REMARK 4.1. If the assumption (4.3) holds for all $L \geqslant 2$, then the solution is infinitely differentiable regarding to the x-variable. Due to the equation (2.5), the solution is C^∞ with respect to both its variables.

PROOF. (Induction on α) For $\alpha = 4$ let u be a solution of (2.5) that satisfies (4.3). The equation (2.5) implies that $u_t \in L^\infty([0,\,T\,];\,L^2(W_{0\,L\,0}))$, where

$$u \in L^\infty([0,\,T];\,H^3(W_{0\,L\,0})), \quad u_t \in L^\infty([0,\,T];\,L^2(W_{0\,L\,0})).$$

Then $u \in C([0,\,T];\,L^2(W_{0\,L\,0})) \bigcap C_w([0,\,T];\,H^3(W_{0\,L\,0}))$. Hence, $u\colon [0,\,T] \mapsto H^3(W_{0\,L\,0})$ is a weakly continuous function. In particular, $u(\,\cdot\,,\,t) \in H^3(W_{0\,L\,0})$ for all t. Let $t_0 \in (0,\,T)$ and $u(\,\cdot\,,\,t_0) \in H^3(W_{0\,L\,0})$. Then, there are $\{\varphi^{(n)}\} \subseteq C_0^\infty(\mathbb{R})$, such that $\varphi^{(n)}(\,\cdot\,) \to u(\,\cdot\,,\,t_0)$ in $H^3(W_{0\,L\,0})$. Let $u^{(n)}(x,\,t)$ be a unique solution of (2.5) with $u^{(n)}(x,\,t_0) = \varphi^{(n)}(x)$. Then, by Theorems 2.1 and 2.2 there exists a unique solution u of (2.5) $u^{(n)} \in L^\infty([t_0,\,t_0+\delta];\,H^3(W_{0\,L\,0}))$ with $u^{(n)}(x,\,t_0) \equiv \varphi^{(n)}(x) \to u(x,\,t_0) \equiv \varphi(x)$ in $H^3(W_{0\,L\,0})$, on a time interval $[t_0,\,t_0+\delta]$, where $\delta > 0$ does not depend on n. Now, by Theorem 2.3, we have $u^{(n)} \in L^\infty([t_0,\,t_0+\delta];\,H^3(W_{0\,L\,0})) \bigcap L^2([t_0,\,t_0+\delta];\,H^4(W_{\sigma,\,L-1,\,0}))$ with a bound that depends only on the norm of $\varphi^{(n)}$ in $H^3(W_{0\,L\,0})$. Furthermore, Theorem 2.3 guarantees the non-uniform bounds

$$\sup_{[t_0,\, t_0+\delta]} \sup_x (1+|x_+|)^k \,|\, \partial^\alpha u^{(n)}(x,\, t)| < +\infty$$

for each n, k and α. The main inequality (3.1) and the estimate (4.2) are, therefore, valid for each $u^{(n)}$ in the interval $[t_0,\, t_0+\delta]$. η may be chosen arbitrarily in its weight class (4.1), and then, ξ is defined by (3.4) and the constant c_1, c_2, c_3, c_4 are independent of n. From (3.1) and (4.2) we have that

$$\sup_{[t_0,\, t_0+\delta]} \int_{\mathbb{R}} \xi [u_\alpha^{(n)}]^2 dx + \int_{t_0}^{t_0+\delta} \int_{\mathbb{R}} \eta [u_{\alpha+1}^{(n)}]^2 dx dt \leqslant c, \tag{4.4}$$

where c is independent of n by (4.2). The estimate (4.4) is proved by induction for $\alpha = 4,\, 5,\, 6,\, \ldots$ Thus, $u^{(n)}$ is also bounded in

$$L^\infty([t_0,\, t_0+\delta];\, H^\alpha(W_{\sigma,\, L-\alpha+3,\, \alpha-3})) \bigcap L^2([t_0,\, t_0+\delta];\, H^{\alpha+1}(W_{\sigma,\, L-\alpha+2,\, \alpha-3})) \tag{4.5}$$

for $\alpha \geqslant 4$. We have $u^{(n)} \to u$ in $L^\infty([t_0,\, t_0+\delta];\, H^3(W_{0\,L\,0}))$. By Corollary 2.9 it follows that u belongs to the space (4.5). Since δ is fixed and t_0 is arbitrary chosen from the interval $(0, T)$, this result is valid over the entire interval $[0,\, T]$. □

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