SCIENTIA
Series A: *Mathematical Sciences*, Vol. 9 (2003), 33–43
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Long-pin perturbations of the trivial solution for Hele-Shaw corner flows

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ABSTRACT. We consider two-dimensional Hele-Shaw corner flows without effect of the surface tension and with an interface extending to the infinity along one of the walls. Explicit solutions that present a "long-pin" deformations of the trivial solution are got. Making use of the Polubarinova-Galin approach we derive parametric equations for the moving interface in terms of univalent mappings of a canonical domain. For the right angle we repeat a result by Leo P. Kadanoff.

1. Introduction

The Hele-Shaw problem involves two inmiscible fluids that interact in a narrow gap between two parallel plates. One of them is of higher viscosity and the other is effectively inviscid. We limit the process by two non-parallel walls that form a corner at infinity (see Figure 1). Neglecting gravity and effect of surface tension we assume that the unique force that act in our system is a homogeneous sink (or source) at infinity.

In 1898 H. S. Hele-Shaw [**12, 13**] first proposed to study a cell named after him. In dimensionless coordinates the moving viscous incompressible fluid is described by a potential flow with a velocity field $\mathbf{V} = (V_1, V_2)$. The pressure p is the potential for the fluid velocity

$$\mathbf{V} = -\frac{h^2}{12\mu}\nabla p,$$

where h is the cell gap and μ is the viscosity of the fluid (see e.g. [**25, 34**]). Through the similarity in the governing equations, Hele-Shaw flows can be used to study the models of saturated flows in porous media. Another typical scenario is given by Witten-Sander's diffusion-limited-aggregation (DLA) model. In both cases the motion takes place in a Laplacian field (pressure for viscous fluid and walker's probability of visit for DLA).

2000 *Mathematics Subject Classification.* Primary 30C45, 76S05, 76D27. Secondary 35Q35, 30C35.

Key words and phrases. free boundary problem, conformal map, complex analysis, univalent function, hypergeometric function.

This work is partially supported by projects: Fondecyt (Chile) #1030373, #1020067, and UTFSM #12.03.23.

Over the years various particular cases of such a flow have been considered. The construction of explicit solutions is an important part of the study of Hele-Shaw flows. A simple non-trivial solution that described the motion of the contact interface was given by P. Ya. Polubarinova-Kochina [**27, 28**] and L. A. Galin [**10**] who suggested in 1945 a complex variable approach that now is one of the basic tools for investigating the Hele-Shaw flow. Another important contribution has been made by P. G. Saffman, G. I. Taylor in 1958 [**34**] who discovered the long time existence of a continuum set of long bubbles between two parallel walls in a Hele-Shaw cell that further have been called the Saffman-Taylor fingers. Curiously, that time Saffman-Taylor's work has been underestimated. For example, a MR review says that "...the authors' analysis does not seem to be completely rigorous, mathematically. Many details are lacking. Besides, the authors do not seem to be aware of the fact that there exists a vast amount of literature concerning viscous fluid flow into porous (homogeneous and non-homogeneous) media in Russian and Romanian". Nowadays, the Saffman-Taylor fingers are widely known in many fields of mechanics, chemistry, and industry. Handling these first steps many

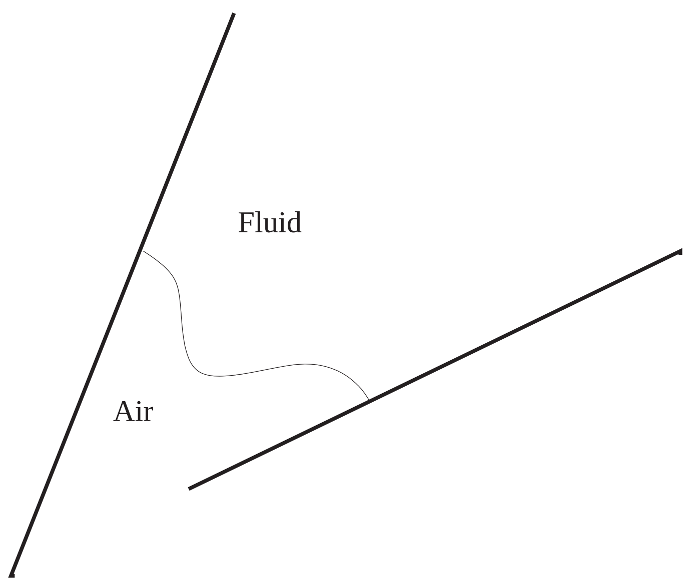

FIGURE 1. Infinite corner

authors have been constructing non-trivial solutions. We should say that mostly these explicit solutions were either polynomials and rational functions, or else, logarithmic solutions linked to Saffman-Taylor fingers. Another type of explicit solutions has been proposed by, e.g., S. Howison, J. King [**19**], L. Cummings [**8**], where the problem was reduced to solving the Poisson equation eliminating time by applying the Baiocchi transformation. The solutions were given making use of the Riemann P-function and hypergeometric functions.

Corner flows of an inviscid incompressible fluid have been studied intensively, e.g., in [**21, 22, 24, 26, 36**] (see also the references therein). In particular, we mention here papers [**2, 3, 4, 5, 20, 32, 33, 35**]. The solution constructed by Leo P. Kadanoff [**20**] is directly linked with our research.

Starting with Polubarinova-Kochina's and Galin's works the usual approach to a description of the motion of the free interface between a viscous fluid and an inviscid one (gas) is to construct an auxiliary time dependent conformal map $z = f(\zeta, t)$ from one of the relevant canonical domains (the unit disk, a half-plane, or a strip) onto the phase domain $\Omega(t)$ occupied by the given viscous fluid.

In our report we will construct explicit solutions in an infinite corner of arbitrary angle such that the viscous fluid is glued to one of the walls and the interface extends to infinity along it and the interface has contact angle $\pi/2$ at a moving contact point at the other wall. These solution will present a long-pin deformation of the trivial (circular) solution. In the case of the right angle we obtain Kadanoff's solution [**20**]. Finally, we present analogous solution in the corner with a source at its vertex.

2. Mathematical model

In this section we deal with a general case of corner flows. We suppose that the viscous fluid occupies a simply connected domain $\Omega(t)$ in the phase z-plane whose boundary $\Gamma(t)$ consists of two walls $\Gamma_1(t)$ and $\Gamma_2(t)$ of the corner and a free interface $\Gamma_3(t)$ between them at a moment t. The inviscid fluid (or air) fills the complement to $\Omega(t)$. The simplifying assumption of constant pressure at the interface between the fluids means that we omit the effect of surface tension. The velocity must be bounded close to the contact point that yields the contact angle between the walls of the corner and the moving interface to be $\pi/2$ (see Figure 2). A limiting case corresponds to one finite contact point and the other tends to infinity. By a shift we can place the point of the intersection of the wall extensions at the origin. To simplify matter, we set the corner of angle α between the walls so that the positive real axis x contains one of the walls and fix this angle as $\alpha \in (0, \pi]$.

Let us mention here that the model can be studied in the presence of surface tension and the macroscopic contact angle between the walls and the free interface can be different from $\pi/2$. Let us denote it by β. The contact angle β at a moving contact line obeys interesting properties that has been studied by R. Ablett [**1**] (see also [**6, 7**]) in a particular case of water in contact with a paraffin surface. It turns out that the steady angle β depends on the velocity of the contact line. The angle β increases with the velocity increased for the liquid advancing over the surface up to a certain value β_0 and, then, remains the same for a greater velocity. Reciprocally, β decreases with the velocity increased for the liquid receding over the surface up to a certain value β_1 (different from β_0) and, then, remains the same for a greater velocity.

In our zero-surface-tension model the field equation for the fluid pressure $p(z,t) \equiv p(x,y,t)$ is simply

$$\Delta p = 0, \quad \text{in the flow region } \Omega(t), \tag{2.1}$$

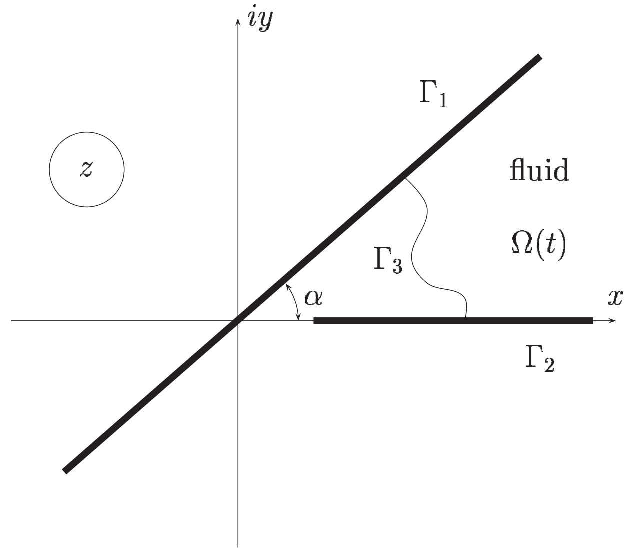

FIGURE 2. $\Omega(t)$ is the phase domain within an infinite corner and the homogeneous sink/source at ∞

and the fluid velocity $\mathbf{V}$ averaged across the gap is $\mathbf{V} = -\nabla p$. The free boundary conditions

$$p\Big|_{\Gamma_3} = 0, \quad \frac{\partial p}{\partial t}\Big|_{\Gamma_3} = (\nabla p)^2 \tag{2.2}$$

are imposed on the free boundary $\Gamma_3 \equiv \Gamma_3(t)$. This implies that the normal velocity v_n of the free boundary Γ_3 outwards from $\Omega(t)$ is given as

$$\frac{\partial p}{\partial n}\Big|_{\Gamma_3} = -v_n. \tag{2.3}$$

On the walls $\Gamma_1 \equiv \Gamma_1(t)$ and $\Gamma_2 \equiv \Gamma_2(t)$ the boundary conditions are given as

$$\frac{\partial p}{\partial n}\Big|_{\Gamma_1 \cup \Gamma_2} = 0. \tag{2.4}$$

We suppose that the motion is driven by a homogeneous source/sink at infinity. Since the angle between the walls at infinity is also α, the pressure behaves about infinity as

$$p \sim \frac{-Q}{\alpha} \log |z|, \quad \text{as } |z| \to \infty,$$

where Q corresponds to the constant strength of the source ($Q < 0$) or sink ($Q > 0$). Finally, we assume that $\Gamma_3(0)$ is a given analytic curve.

We introduce a complex analytic potential $W(z,t) = p(z,t) + i\psi(z,t)$, where $-\psi$ is the stream function. Then, $\nabla p = \overline{\partial W/\partial z}$ by the Cauchy-Riemann conditions. Let us consider an auxiliary parametric complex ζ-plane, $\zeta = \xi + i\eta$. We set $D = \{\zeta : |\zeta| > 1,\ 0 < \arg \zeta < \alpha\}$, $D_3 = \{z : z = e^{i\theta},\ \theta \in (0,\alpha)\}$, $D_1 = \{z : z = re^{i\alpha},\ r > 1\}$,

$D_2 = \{z : z = r, r > 1\}$, $\partial D = D_1 \cup D_2 \cup D_3$, and construct a conformal univalent time-dependent map $z = f(\zeta, t)$, $f : D \to \Omega(t)$, so that being continued onto ∂D, $f(\infty, t) \equiv \infty$, and the circular arc D_3 of ∂D is mapped onto Γ_3 (see Figure 3). This

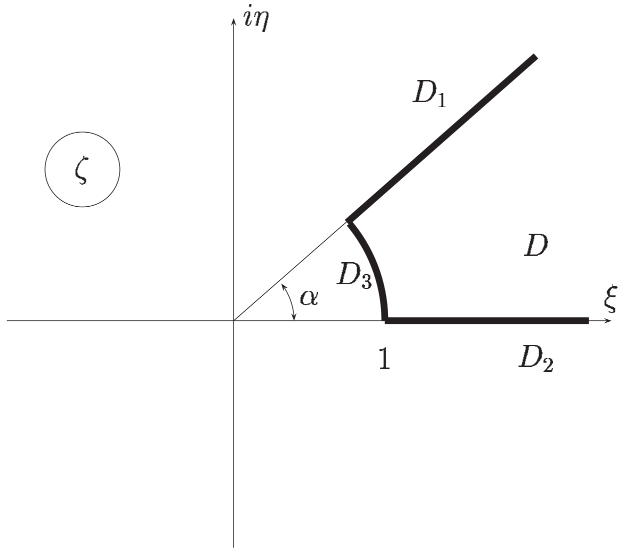

FIGURE 3. The parametric domain D

map has an expansion

$$f(\zeta, t) = \zeta \sum_{n=0}^{\infty} a_n(t) \zeta^{-\frac{\pi n}{\alpha}}$$

about infinity and $a_0(t) > 0$. The function f parameterizes the boundary of the domain $\Omega(t)$ by $\Gamma_j = \{z : z = f(\zeta, t), \zeta \in D_j\}$, $j = 1, 2, 3$.

We will use the notations $\dot{f} = \partial f / \partial t$, $f' = \partial f / \partial \zeta$. The normal unit vector in the outward direction is given by

$$\hat{n} = -\zeta \frac{f'}{|f'|} \text{ on } \Gamma_3, \;\; \hat{n} = -i \text{ on } \Gamma_2, \text{ and } \;\hat{n} = ie^{i\alpha} \text{ on } \Gamma_1.$$

Therefore, the normal velocity is obtained as

$$v_n = \mathbf{V} \cdot \hat{n} = -\frac{\partial p}{\partial n} = \begin{cases} -\mathrm{Re} \left(\dfrac{\partial W}{\partial z} \dfrac{\zeta f'}{|f'|} \right), & \text{for } \zeta \in D_3 \\ 0, & \text{for } \zeta \in D_1 \\ 0, & \text{for } \zeta \in D_2 \end{cases} \tag{2.5}$$

The superposition $W \circ f$ is the solution to the mixed boundary problem (2.1), (2.2), (2.4) in D, therefore, it is the Robin function given by $W \circ f = -\frac{Q}{\alpha} \log \zeta$. On the other hand,

$$v_n = \begin{cases} \text{Re } (\dot{f}\overline{\zeta f'}/|f'|), & \text{for } \zeta \in D_3 \\ \text{Im } (\dot{f}e^{-i\alpha}), & \text{for } \zeta \in D_1 \\ -\text{Im } (\dot{f}), & \text{for } \zeta \in D_2 \end{cases} \tag{2.6}$$

The first lines of (2.5), (2.6) give us that

$$\text{Re } (\dot{f}\ \overline{\zeta f'}) = \frac{Q}{\alpha}, \quad \text{for } \zeta \in D_3. \tag{2.7}$$

The resting lines of (2.5), (2.6) imply

$$\text{Im } (\dot{f}e^{-i\alpha}) = 0 \quad \text{for } \zeta \in D_1, \quad \text{Im } (\dot{f}) = 0 \quad \text{for } \zeta \in D_2. \tag{2.8}$$

One of the typical properties of the problem (2.1–2.4) is that starting with an analytic boundary component $\Gamma_3(0)$, the one-parameter evolutionary chain of solutions develops possible cusps at a finite blow-up time t^*. Another typical scenario is fingering. It is known that the weak solution exists locally in time which is even coincides with the classical one in the case of an analytic boundary Γ_3. The development of these topics exceed the scope of our paper. We only refer the reader to some relevant works [**9, 16, 18, 29, 31, 34**].

Let us make some relevant comments about the geometry of the bubbles. In [**15, 30, 37, 38**] we studied geometric properties of the moving interface that are preserved during the time of existence of solutions to the Hele-Shaw problem in the stable case (well-posed problems). Here we note that all considerations of the mentioned works can be applied in our case. In particular, for the advancing fluid in the corner, the interface is star-shaped during the time of evolution if the initial interface is. In [**23**] we found explicit self-similar solutions in an arbitrary corner with an extending bubble. Continuing this work we found here explicit solutions that present long-pin deformations of the trivial solution.

3. Explicit long-pin solutions

We consider the case of a sink at infinity ($Q > 0$). The simplest explicit solution in the above model is

$$f(\zeta, t) = \sqrt{\frac{2Qt}{\alpha}}\zeta,$$

that produces a circular dynamics of the free boundary. We are aimed at a perturbation of this trivial solution by a function independent of t, say we are looking for the solution in the form

$$f(\zeta, t) = \sqrt{\frac{2Qt}{\alpha}}\zeta + \zeta g(\zeta),$$

where $g(\zeta)$ is regular in D with the expansion

$$g(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{\zeta^{\frac{\pi n}{\alpha}}}$$

about infinity. The branch is chosen so that g, being continued symmetrically into the reflection of D is real at real points. The equation (2.7) implies that on D_3 the function g satisfies the equation

$$\mathrm{Re}\,(g(\zeta)+\zeta g'(\zeta))=0,\quad \zeta\in D_3.$$

Taking into account the expansion of g we are looking for a solution satisfying the equation

$$g(\zeta)+\zeta g'(\zeta)=\frac{\zeta^{\frac{\pi}{\alpha}}-1}{\zeta^{\frac{\pi}{\alpha}}+1},\quad \zeta\in D. \tag{3.1}$$

Changing the right-hand side of the above equation one would obtain other solutions. The general solution to (3.1) can be given in terms of the Gauss hypergeometric function $\mathbf{F}\equiv_2\mathbf{F}_1$ as

$$\zeta g(\zeta)=\zeta-2\zeta\mathbf{F}\left(\frac{\alpha}{\pi},1,1+\frac{\alpha}{\pi};-\zeta^{\frac{\pi}{\alpha}}\right)+C.$$

We note that f' vanishes for $\zeta^{\frac{\pi}{\alpha}}=(2/(1+\sqrt{2Qt/\alpha}))-1$, therefore, the function f is locally univalent, the cusp problem is degenerating and appears only at the initial time $t=0$ and the solution exists during infinite time. The resulting function is homeomorphic on the boundary ∂D, hence it is univalent in D. This presents a case (apart from the trivial one) of the long existence of the solution in the problem with suction (ill-posed problem). To complete our solution we need to determine the constant C. First of all we choose the branch of the function ${}_2\mathbf{F}_1$ so that the points of the ray $\zeta>1$ have real images. This implies that $\mathrm{Im}\,C=0$. We continue verifying the asymptotic properties of the function $f(e^{i\theta},t)$ as $\theta\to\alpha-0$. The slope is

$$\lim_{\theta\to\alpha-0}\arg[ie^{i\theta}f'(e^{i\theta},t)]=\alpha+\frac{\pi}{2}+\lim_{\theta\to\alpha-0}\arg\left(\sqrt{\frac{2Qt}{\alpha}}+\frac{e^{i\frac{\pi\theta}{\alpha}}-1}{e^{i\frac{\pi\theta}{\alpha}}+1}\right)=\alpha+\pi.$$

To calculate shift we choose C such that

$$\lim_{\theta\to\alpha-0}\mathrm{Im}\,[e^{-i\alpha}f'(e^{i\theta},t)]=0.$$

Using the properties of hypergeometric functions we have

$$\lim_{\gamma\to0+0}\mathrm{Im}\,\mathbf{F}\left(\frac{\alpha}{\pi},1,1+\frac{\alpha}{\pi};e^{i\gamma}\right)=\frac{\alpha}{2}.$$

Therefore, $C=\alpha$. We present numerical simulation in Figure 4.

The special case of angle $\alpha=\pi/2$ has been considered by Leo P. Kadanoff [**20**]. The hypergeometric function is reduced to arctangent and we obtain

$$f(\zeta,t)=(\sqrt{4Qt/\pi}+1)\zeta+i\log\frac{1+i\zeta}{1-i\zeta}+\frac{\pi}{2},\quad Q>0.$$

This function maps the domain $\{|\zeta|>1,\ 0<\arg\zeta<\pi/2\}$ onto an infinite domain bounded by the imaginary axis (Γ_1), the ray $\Gamma_2=\{r:\,r\geqslant\sqrt{4Qt/\pi}+1\}$ of the real axis and an analytic curve Γ_3 which is the image of the circular arc, see Figure 5.

By the analogy with the infinite sink we are able to give long-pin solutions for a finite source (see Figure 6). The phase domain is a simply connected finite domain at the vertex of the corner which is a source. We locate the corner so that one of the

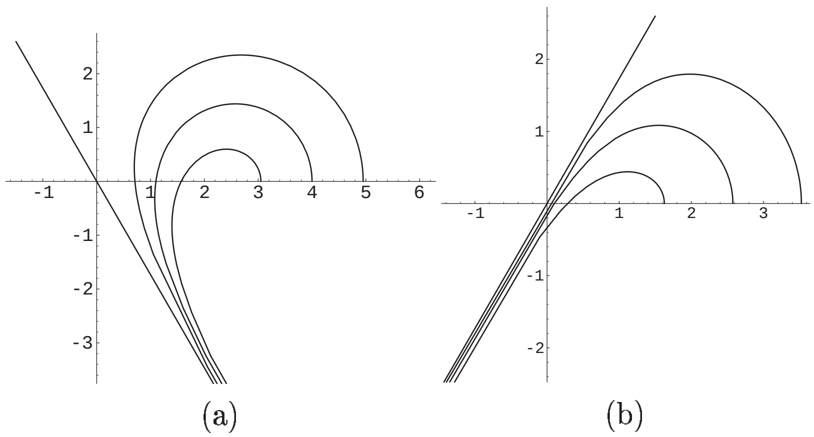

FIGURE 4. Long-pin dynamics in the corner of angle: (a) $\alpha = 2\pi/3$; (b) $\alpha = \pi/3$

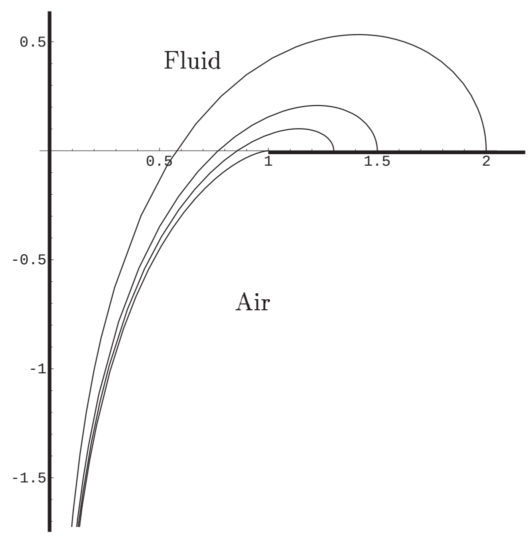

FIGURE 5. Kadanoff's solution

walls lie on the real axis and the other forms the corner of angle α at the origin. We set $G = \{\zeta : |\zeta| < 1,\, 0 < \arg\zeta < \alpha\}$, $G_3 = \{z : z = e^{i\theta},\, \theta \in (0,\alpha)\}$, $G_1 = \{z : z = re^{i\alpha},\, r < 1\}$, $G_2 = \{z : z = r,\, r < 1\}$, $\partial G = G_1 \cup G_2 \cup G_3$, and construct a conformal univalent time-dependent map $z = f(\zeta, t)$, $f : G \to \Omega(t)$. This map has an expansion

$$f(\zeta, t) = \zeta \sum_{n=0}^{\infty} a_n(t) \zeta^{\frac{\pi n}{\alpha}}$$

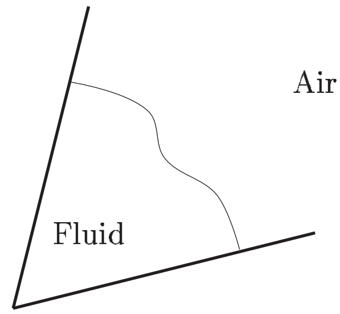

FIGURE 6. Finite source

about the origin and $a_0(t) > 0$. The equations for this function at the boundary of G are

$$\mathrm{Re}\ (\dot{f}\ \overline{\zeta f'}) = -\frac{Q}{\alpha}, \quad \text{for } \zeta \in G_3,$$

where $Q < 0$, and

$$\mathrm{Im}\ (\dot{f} e^{-i\alpha}) = 0 \quad \text{for } \zeta \in G_1, \quad \mathrm{Im}\ (\dot{f}) = 0 \quad \text{for } \zeta \in G_2.$$

We give a solution analogous to the infinite case by

$$f(\zeta, t) = \sqrt{\frac{2|Q|t}{\alpha}}\zeta - \zeta + 2\zeta \mathbf{F}\left(\frac{\alpha}{\pi}, 1, 1 + \frac{\alpha}{\pi}; -\zeta^{\frac{\pi}{\alpha}}\right)$$

The numerical simulation is presented in Figure 7

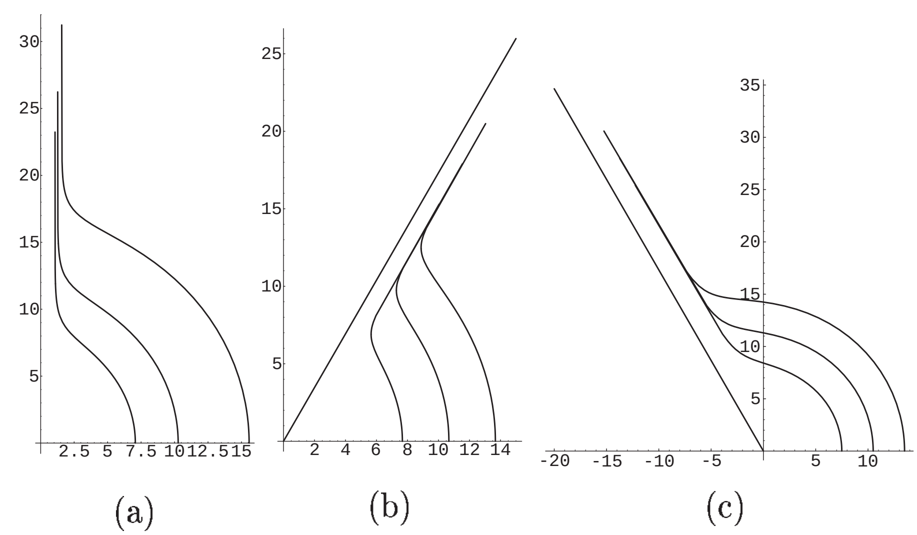

FIGURE 7. Long-pin dynamics of the advancing fluid in the corner of angle: (a) $\alpha = \pi/2$; (b) $\alpha = \pi/3$; (c) $\alpha = 2\pi/3$

Remark. By the proposed method we can perturbate various known self-similar solutions even for more general flows. The idea is as follows. Let $f_0(\zeta,t)=H(t)F(\zeta)$ be a known solution to the problem, the basic equation of which is the Polubarinova-Galin equation $\mathrm{Re}\,(\dot f\,\overline{\zeta f'})=Q$ (for positive or negative Q) where ζ belongs to a circular component of the parametric domain. We are looking for a new solution in the form $f(\zeta,t)=f_0(\zeta,t)+g(\zeta)$, where $g(\zeta)$ is an analytic function with an appropriate expansion. Then, on the circular component this function satisfies the equation

$$\mathrm{Re}\,\frac{\zeta g'(\zeta)}{F(\zeta)}=0.$$

So one must solve the equation $\zeta g'(\zeta)=F(\zeta)P(\zeta)$, where $P(\zeta)$ is a function with the vanishing real part at the points of the circular component.

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Received 15 03 2003

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