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Fourth coefficient estimate in the class of univalent functions with quasiconformal extensions

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Abstract. We denote by $S(k)$ the class of all univalent conformal maps f defined in the unit disk Δ normalized by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, such that all f admit k-quasiconformal homeomorphic extension to the whole Riemann sphere $\hat{\mathbb{C}}$, and $f(\infty) = \infty$. In our note we give a new estimate for $|a_4|$ in $S(k)$ making use of the Area Principle.

Let us denote by Σ the class of functions

$$F(\zeta) = \zeta + \alpha_0 + \frac{\alpha_1}{\zeta} + \dots,$$

which are regular and univalent in the exterior part of the unit disk $\Delta' = \{\zeta : |\zeta| > 1\}$ except for the simple pole at infinity and its subclass Σ_0 is given by an additional restriction $0 \notin F(\Delta')$. Let $\Sigma(k)$ stand for the subclass of functions $F \in \Sigma$ that admit k-quasiconformal homeomorphic extensions to the unit disk Δ, and $\Sigma_0(k)$ be obtained from $\Sigma(k)$ applying the restriction $F(0) = 0$. By $S(k)$ we denote the class of all univalent conformal maps f defined in the unit disk Δ normalized by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, such that all f admit k-quasiconformal homeomorphic extensions to the whole Riemann sphere $\hat{\mathbb{C}}$, and $f(\infty) = \infty$. Obviously, $f \in S(k)$ if and only if $1/f(1/\zeta) \in \Sigma_0(k)$. During the long history of univalent functions the Bieberbach Conjecture [**1**] $|a_n| \leqslant n$, $f \in S$, has been the most intriguing one. It has been proved by L. de Branges in 1984 [**2, 3**]. In spite of many works about coefficient estimates in the class S, there are some difficult problems that still unsolved, in particular, the problem of estimating $|a_n|$, $n > 2$, for the subclass $S(k)$ that we will deal with for $n = 4$. We remark here that the only known complete sharp estimate is $|a_2| \leqslant 2k$, $0 \leqslant k < 1$. Some achievements in this estimate are as follows. S. L. Krushkal and R. Kühnau [**6**] gave the estimate

$$|a_4| \leqslant \frac{2}{3}k + O(k^4) < \frac{2}{3}k + \frac{14}{3}k^4,$$

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as $k \to 0$ in the class $S(k)$. In [**4**] it was shown that

$$|a_4| \leqslant \begin{cases} \frac{2}{3}k + \frac{4}{\sqrt{3}}k^2 + \frac{10}{3}k^3, & \text{for } 0 \leqslant k < \frac{\sqrt{7}}{15}, \\ \frac{2}{3}k + \frac{10}{3}k^3, & \text{for } \frac{\sqrt{7}}{15} \leqslant k < 1, \end{cases}$$

that leads to $|a_4| \leqslant 4$ as $k \to 1$. In 1995 S. L. Krushkal [**7**] obtained the sharp estimate

$$|a_n| \leqslant \frac{2k}{n-1}, \quad f \in S(k),$$

under the restriction

$$0 < k \leqslant \frac{1}{n^2+1}, \quad n > 2,$$

that implies $|a_4| \leqslant 2k/3$ for $0 < k \leqslant 1/17 = 0.0588\ldots$. V. G. Sheretov [**9**] gave some general conditions for the coefficients of p-symmetric univalent functions from $S(k)$ and $\Sigma_0(k)$.

In our note we use the Area Principle (see, e.g., [**8**]) to give a estimate for $|a_4|$ for functions from $S(k)$. Our result is the following theorem.

THEOREM 0.1. *In the class* $S(k)$ *we have*

$$|a_4| \leqslant \frac{2}{3}k + \frac{2}{3}k\gamma(x^*), \quad \textit{for} \quad 0.15 \leqslant k \leqslant \tfrac{\sqrt{7}}{15}\ ,$$

where x^* *is a unique root of the equation*

$$3(0.22 - k^2)x^2 - 3.68x + 6k^2 + 1.62 = 0, \ x^* \in (0,1),$$

and the function $\gamma(x)$ *is given as*

$$\gamma(x) = (0.22 - k^2)x^3 - 1.84x^2 + (6k^2 + 1.62)x.$$

PROOF. Let us follow a method by V. Ya. Gutlyanskiĭ [5]. If $F(\zeta) \in \Sigma(k)$, and $Q(w) \neq const$ is a function which is regular in the domain $D_\rho(F) = F(\Delta'_\rho)$, where $\Delta'_\rho = \{\zeta : 1 < \rho \leqslant |\zeta|\}$, then one obtains the Laurent series of the function $Q(F(\zeta))$ in the annulus $1 < |\zeta| < \rho$ as

$$Q(F(\zeta)) = \sum_{n=1}^{\infty} \omega_n \zeta^{-n} + \sum_{n=0}^{\infty} \gamma_n \zeta^n.$$

Using the Area Principle we obtain

$$\sum_{n=1}^{\infty} n|\omega_n|^2 \leqslant k^2 \sum_{n=1}^{\infty} n|\gamma_n|^2. \tag{0.1}$$

For arbitrary constants $x_p, x'_p,\ p = 1, 2, \ldots$, such that

$$0 < \sum_{p=1}^{\infty} \frac{|x_p|^2}{p} < \infty, \ 0 < \sum_{p=1}^{\infty} \frac{|x'_p|^2}{p} < \infty,$$

the inequality (0.1) implies

$$\sum_{q=1}^{\infty} q \Big| \sum_{p=1}^{\infty} \omega_{p,q} x_p x'_q \Big|^2 \leqslant k^2 \sum_{p=1}^{\infty} \frac{|x_p|^2}{p} \sum_{q=1}^{\infty} \frac{|x'_q|^2}{q}, \tag{0.2}$$

where $\omega_{p,q}$ are the Grunsky coefficients. We assume $x_p = x'_q$ in (0.2), and consider the subclass $\Sigma^2(k)$ of odd functions F from $\Sigma(k)$. For our convenience we leave the notations $\omega_{p,q}$, and from (0.2) it follows that

$$\sum_{q=1}^{\infty}(2q-1)|\sum_{p=1}^{\infty}\omega_{2p-1,2q-1}x_p|^2 \leqslant k^2\sum_{p=1}^{\infty}\frac{|x_p|^2}{2p-1}. \tag{0.3}$$

First, we assume $x_1 = 1,\ x_p = 0,\ p = 2,\ldots,$ and choose $x_1 = l,\ x_2 = 2,\ x_p = 0,\ p = 3,\ldots.$ Then we have

$$|\omega_{1,1}|^2 + 3|\omega_{1,3}|^2 \leqslant k^2, \tag{0.4}$$

$$|\omega_{1,1}l + 2\omega_{1,3}|^2 + 3|\omega_{1,3}l + 2\omega_{3,3}|^2 \leqslant k^2(|l|^2 + \frac{4}{3}). \tag{0.5}$$

One easily sees (e.g.,[**8**]) that

$$\omega_{3,3} = \frac{a_4}{2} - 4\omega_{1,1}\omega_{1,3} - 5\frac{\omega_{1,1}^3}{3}, \quad \omega_{1,1} = \frac{a_2}{2}.$$

Substituting $\omega_{3,3}$ in (0.5) we have

$$3|a_4 - (8\omega_{1,1} - l)\omega_{1,3} - \frac{10}{3}\omega_{1,1}^3|^2 + |\omega_{1,1}l + 2\omega_{1,3}|^2 \leqslant k^2(|l|^2 + \frac{4}{3}). \tag{0.6}$$

Without loss of generality, we assume $a_4 > 0$. Changing in the left-hand side the absolute value by the real part we get

$$a_4 \leqslant \frac{2}{3}k + \frac{1}{4}|l|^2(k - \frac{1}{k}|\omega_{1,1}|^2) - \frac{1}{k}|\omega_{1,3}|^2 + \mathrm{Re}\ \{(8\omega_{1,1} - l - \frac{1}{k}\bar{\omega}_{1,1}\bar{l})\omega_{1,3} + \frac{10}{3}\omega_{1,1}^3\}.$$

We introduce the following notations (see, e.g., [**8**])

$$\omega_{1,1} = kxe^{i\varphi},\ (x = \frac{|a_2|}{2k}), (0 \leqslant x \leqslant 1), \quad l = \frac{8kx}{1+x}e^{-i\varphi/2}\cos\frac{3}{2}\varphi,\ y = |\sin\frac{3}{2}\varphi|.$$

Then,

$$\begin{aligned}
|a_4| \quad &\leqslant \quad \frac{2}{3}k + 16\frac{k^3x^2}{(1+x)^2}(1-x^2) - 16\frac{k^3x^2}{(1+x)^2}y^2(1-x^2) - \frac{1}{k}|\omega_{1,3}|^2 \\
&+ \quad \mathrm{Re}\ [(8kxe^{i\varphi} - 8kxe^{-i\varphi/2}\cos\frac{3}{2}x)\omega_{1,3} + \frac{10}{3}k^3x^3e^{3i\varphi}] \\
&\leqslant \quad \frac{2}{3}k + 16k^3x^2\frac{1-x}{1+x} - (\frac{20}{3}k^3x^3 + 16\frac{k^3x^2(1-x)}{1+x})y^2 \\
&- \quad \frac{1}{k}|\omega_{1,3}|^2 + 8kx|\omega_{1,3}|y + \frac{10}{3}k^3x^3.
\end{aligned}$$

Now we set the function $q(y)$ by

$$q(y) = -(\frac{20}{3}k^3x^3 + 16\frac{k^3x^2(1-x)}{1+x})y^2 + 8kx|\omega_{1,3}|y.$$

It is easily seen that

$$\max_{0\leqslant y\leqslant 1} q(y) = q(y_0),$$

where

$$y_0 = \frac{kx|\omega_{1,3}|}{\frac{5}{3}k^3x^3 + 4\frac{k^3x^2(1-x)}{1+x}},$$

$|\omega_{1,3}|^2 \leqslant \frac{1-x^2}{3}k^2$ (see (0.4)).

So the estimate of $|a_4|$ is of the form

$$|a_4| \leqslant \frac{2}{3}k + 16k^3x^2\frac{1-x}{1+x} + \frac{10}{3}k^3x^3 + \frac{1}{3}kx\frac{19+14x-5x^2}{12-7x+5x^2}(1-x), x \in (0,1).$$

We note that

$$\frac{2x}{1+x} \leqslant \frac{1+x}{x},\ \frac{19+14x-5x^2}{12-7x+5x^2} \leqslant 3.24 - 0.44x, 0 \leqslant x \leqslant 1.$$

Then,

$$|a_4| \leqslant \frac{2}{3}k + \frac{2}{3}k[(6k^2+1.62)x - 1.84x^2 + (0.22-k^2)x^3] = \frac{2}{3}k + \frac{2}{3}k\gamma(x).$$

Calculating $\gamma'(x)$,we have

$$\gamma'(x) = 3(0.22-k^2)x^2 - 3.68x + 6k^2 + 1.62.$$

Then, the equation $\gamma'(x) = 0$ has a unique solution x^* in (0,1), and correspondingly, $\gamma(x)$ has a unique maximum in (0,1) for $k^2 < 0.22$:

$$\max_{0 \leqslant x \leqslant 1} \gamma(x) = \gamma(x_*).$$

It is easily seen that

$$0 < x < 1, \gamma(0) = 0,\ \gamma(1) = 5k^2,$$

what completes theorem. □

Remarks.

(1) Some achievements in our estimate are as follows. R. Kühnau [**4**] gave the estimate

$$|a_4| \leqslant \frac{2}{3}k + \frac{4}{\sqrt{3}}k^2 + \frac{10}{3}k^3,\ 0 < k < \frac{\sqrt{7}}{15}.$$

Assuming $x^* = x^*(k)$ we have $\gamma(x) = \gamma(x^*(k)) = \gamma_1(k)$ and the function $\frac{2}{3}k\gamma_1(k)$ increases.

We note that

$$\psi(k) = \frac{4}{\sqrt{3}}k^2 + \frac{10}{3}k^3,\ \psi(0.15) > \frac{2}{3}k\gamma_1(\frac{\sqrt{7}}{15}).$$

Then,

$$\psi(k) > \frac{2}{3}k\gamma_1(k), k \in [0.15; \frac{\sqrt{7}}{15}].$$

By more precise calculations, the segment $[0.15; \frac{\sqrt{7}}{15}]$ could be improved up to $[0.1013, \frac{\sqrt{7}}{15}]$.

(2) Let $x = \frac{|a_2|}{2k}$. Then

$$|a_4| \leqslant \frac{2}{3}k + \frac{2}{3}k[(6k^2 + 1.62)\frac{|a_2|}{2k} - 1.84\frac{|a_2|^2}{4k^2} + (0.22 - k^2)\frac{|a_2|^3}{8k^3}].$$

Therefore, we obtaine the sharp estimate under the restriction $a_2 = 0$, and the extremal function is

$$f(z) = z(1 - k\eta z)^{-2/3},\ 0 < k < 1,\ |\eta| = 1.$$

(3) If $k^2 \geqslant 0.22$, we have the estimate [**4**] $|a_4| \leqslant \frac{2}{3}k + \frac{10}{3}k^3$.

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